Sum, Difference, Product, Quotient and Composition of Functions

Sum: $(f+g)(x) = f(x) + g(x)$

Difference: $(f-g)(x) = f(x) - g(x)$

Product: $(f \cdot g)(x) = f(x) \cdot g(x)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\quad g(x) \neq 0$

1. For $f(x) = x^2 + 2$ and $g(x) = x - 6$ find:

a) $(f+g)(x) = f(x) + g(x)$

$x^2 + 2 + x - 6$

$\boxed{x^2 + x - 4}$

b) $(f-g)(x) = f(x) - g(x)$

$x^2 + 2 - (x - 6)$

$x^2 + 2 - x + 6$

$\boxed{x^2 - x + 8}$

c) $(f \cdot g)(x) = f(x) \cdot g(x)$

$(x^2 + 2)(x - 6)$

$\boxed{x^3 - 6x^2 + 2x - 12}$

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\quad g(x) \neq 0$

$\boxed{\frac{x^2 + 2}{x - 6}}$

$x - 6 \neq 0$

$+6 \quad +6$

$\boxed{x \neq 6}$

e) $(f+g)(-4) = f(-4) + g(-4)$

$(x^2+2) + (x-6)$

$((-4)^2+2) + (-4-6)$

$(16+2) + (-10)$

$18 + -10 = \boxed{8}$

f) $(f+g)(x-4) = f(x-4) + g(x-4)$

$(x^2+2) + (x-6)$

$((x-4)^2+2) + (x-4-6)$

$(x-4)(x-4)$

$(x^2-4x-4x+16+2) + (x-10)$

$x^2-8x+18+x-10$

$\boxed{x^2-7x+8}$

Composition of Functions

$(f \circ g)(x) = f(g(x))$

$(g \circ f)(x) = g(f(x))$

2. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

a) $f(x) = \sqrt[3]{x+2}, \; g(x) = x^3 - 2$

$(f \circ g)(x) = f(g(x))$

$\sqrt[3]{x^3-2+2}$

$\sqrt[3]{x^3-\cancel{2}+\cancel{2}}$

$\sqrt[3]{x^3}$

$\boxed{x}$

$(g \circ f)(x) = g(f(x))$

$\sqrt[3]{x+2}^{\,3} - 2$

$(\sqrt[3]{x+2})^{\cancel{3}} - 2$

$x + \cancel{2} - \cancel{2}$

$\boxed{x}$

b) $f(x)=\frac{1}{5}x-2,\ \ g(x)=5x+2$

$(f\circ g)(x)=f(g(x))$

$\frac{1}{5}(5x+2)-2$

$x+\frac{2}{5}-2$

$\frac{2}{5}-\frac{2\cdot5}{1\cdot5}$

$\frac{2}{5}-\frac{10}{5}=-\frac{8}{5}$

$\boxed{x-\frac{8}{5}}$

$(g\circ f)(x)=g(f(x))$

$5\left(\frac{1}{5}x-2\right)+2$

$x-10+2$

$\boxed{x-8}$

c) $f(x)=\sqrt{x+1},\ \ g(x)=x^2-3$

$(f\circ g)(x)=f(g(x))$

$\sqrt{(x^2-3)+1}$

$\sqrt{x^2-3+1}$

$\boxed{\sqrt{x^2-2}}$

$(g\circ f)(x)=g(f(x))$

$\left(\sqrt{x+1}\right)^2-3$

$x+1-3$

$\boxed{x-2}$

d) $f(x)=\sqrt{x^2-1}$, $g(x)=\dfrac{x^2}{x^2+2}$

$(g\circ f)(x)=g(f(x))$

$$\frac{\left(\sqrt{x^2-1}\right)^2}{\left(\sqrt{x^2-1}\right)^2+2}$$

$$\frac{x^2-1}{x^2-1+2}=\boxed{\frac{x^2-1}{x^2+1}}$$

$(f\circ g)(x)=f(g(x))$

$$\sqrt{\left(\frac{x^2}{x^2+2}\right)^2-1}$$

$$\left(\frac{x^2}{x^2+2}\right)^2=\frac{(x^2)^2}{(x^2+2)^2}$$

$$\frac{x^4}{(x^2+2)(x^2+2)}=\frac{x^4}{x^4+2x^2+2x^2+4}$$

$$\frac{x^4}{x^4+4x^2+4}$$

$$\sqrt{\frac{x^4}{x^4-4x^2+4}-\frac{1}{1}\cdot\frac{x^4-4x^2+4}{x^4-4x^2+4}}$$

LCD $= x^4-4x^2+4$

$$\sqrt{\frac{\cancel{x^4}-\cancel{x^4}+4x^2-4}{x^4-4x^2+4}}$$

$$\sqrt{\frac{4x^2-4}{x^4-4x^2+4}}=\frac{\sqrt{4(x^2-1)}}{\sqrt{(x^2-2)^2}}=\boxed{\frac{2\sqrt{x^2-1}}{|x^2-2|}}$$

3. Find two functions f and g such that $(f\circ g)(x)=h(x)$.

a) $h(x)=\sqrt[3]{2-x}$

$f(g(x))$

$f(g(x))=\sqrt[3]{(2-x)}$

$$\boxed{\begin{array}{l} g(x)=2-x \\ f(x)=\sqrt[3]{x} \end{array}}$$

b) $h(x) = \dfrac{3}{(\underbrace{2x-1}_{})^2}$

$f(g(x)) = \dfrac{3}{(2x-1)^2}$

$$\boxed{\begin{gathered} g(x) = 2x-1 \\ f(x) = \dfrac{3}{x^2} \end{gathered}}$$

c) $h(x) = (\underbrace{x+1}_{})^3 + 2(\underbrace{x+1}_{})^2$

$f(g(x)) = (x+1)^3 + 2(x+1)^2$

$$\boxed{\begin{gathered} g(x) = x+1 \\ f(x) = x^3 + 2x^2 \end{gathered}}$$