Proofs of Derivatives of Trigonometric Functions

To prove the derivatives of trigonometric functions, use the definition of the derivative and the trigonometric identities.

<u>Definition of the Derivative:</u>

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

<u>Special Limits:</u>

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

<u>Trigonometric Identities:</u>

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$
$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$\tan x=\frac{\sin x}{\cos x}$$
$$\sec x=\frac{1}{\cos x}$$
$$\csc x=\frac{1}{\sin x}$$
$$\cot x=\frac{\cos x}{\sin x}$$

<u>Proof:</u> $\frac{d}{dx}\sin x=\cos x$

$f(x)=\sin x$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\overbrace{\sin(x+h)}^{f(x+h)}-\overbrace{\sin x}^{f(x)}}{h}=\frac{\overbrace{\sin x\cos h+\sin h\cos x}^{\sin(x+h)}-\sin x}{h}$$

GCF = $\sin x$

$$\lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}=\lim_{h\to 0}\frac{(\sin x\cos h-\sin x)+\sin h\cos x}{h}$$

$$\lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h}=\lim_{h\to 0}\sin x\underbrace{\frac{(\cos h-1)}{h}}_{=0}+\lim_{h\to 0}\underbrace{\frac{\sin h}{h}}_{=1}\cos x$$

$$=(\sin x)(0)+(1)(\cos x)=\boxed{\cos x}$$

<u>Proof:</u> $\frac{d}{dx}\cos x=-\sin x$

$f(x)=\cos x$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\overbrace{\cos(x+h)}^{f(x+h)}-\overbrace{\cos x}^{f(x)}}{h}=\frac{\overbrace{\cos x\cos h-\sin x\sin h}^{\cos(x+h)}-\cos x}{h}$$

GCF = $\cos x$

$$\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}=\lim_{h\to 0}\frac{(\cos x\cos h-\cos x)-\sin x\sin h}{h}$$

$$\lim \cos x(\cos h-1)-\sin x\sin h=\lim \cos x(\cos h-1)-\lim \sin x\sin h$$

$h \to 0$ $\qquad \dfrac{\qquad}{h}$ $\qquad h \to 0$ $\qquad \underbrace{\dfrac{\quad}{h}}_{=0}$ $\qquad h \to 0$ $\qquad \underbrace{\dfrac{\quad}{h}}_{=1}$

$= (\cos x)(0) - (\sin x)(1) = \boxed{-\sin x}$

Proof: $\dfrac{d}{dx} \tan x = \sec^2 x$

$f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$f'(x) = \dfrac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \dfrac{\overbrace{\cos^2 x + \sin^2 x}^{=1}}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \boxed{\sec^2 x}$

Proof: $\dfrac{d}{dx} \sec x = \sec x \tan x$

$f(x) = \sec x = \dfrac{1}{\cos x}$

$f'(x) = \dfrac{(0)(\cos x) - (1)(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \underbrace{\dfrac{1}{\cos x}}_{\sec x} \cdot \underbrace{\dfrac{\sin x}{\cos x}}_{\tan x} = \boxed{\sec x \tan x}$

Proof: $\dfrac{d}{dx} \csc x = -\csc x \cot x$

$f(x) = \csc x = \dfrac{1}{\sin x}$

$f'(x) = \dfrac{(0)(\sin x) - (1)(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = \underbrace{\dfrac{-1}{\sin x}}_{-\csc x} \cdot \underbrace{\dfrac{\cos x}{\sin x}}_{\cot x} = \boxed{-\csc x \cot x}$

<u>Proof:</u> $\frac{d}{dx}\cot x = -\csc^2 x$

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

$$f'(x) = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1(\overbrace{\sin^2 x + \cos^2 x}^{1})}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = \boxed{-\csc^2 x}$$