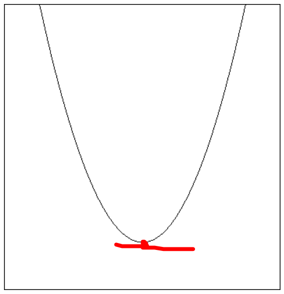
Concavity and the Second Derivative Test

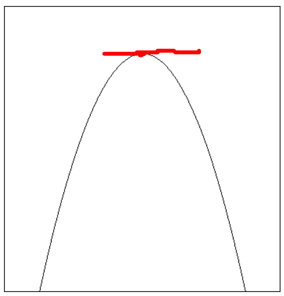
Concavity

A function is concave up on an interval if the graph lies above the horizontal tangent.



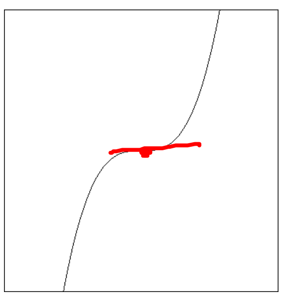
$f''(x_0) > 0$

A function is concave down on an interval if the graph lies below the horizontal tangent.



$f''(x_0) < 0$

Inflection Point - A point on the graph where the concavity changes.



$f''(x_0) = 0$ or $f''(x_0) = DNE$

Second Derivative Test

Let x_0 be a critical point.

If $f'(x_0) = 0$ and $f''(x_0) < 0$ then $(x_0, f(x_0))$ is a relative maximum.

If $f'(x_0) = 0$ and $f''(x_0) > 0$ then $(x_0, f(x_0))$ is a relative minimum.

If $f'(x_0) = 0$ and $f''(x_0) = 0$ then the test fails.

1. Find the points of inflection and determine the concavity of the graph of the function.

a) $f(x)=2x^4-8x+3$

$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2$

$f''(x) = 0$

$24x^2 = 0$

$x = 0$

$f(0) = 2(0)^4 - 8(0) + 3$

$f(0) = 3$

$(0, 3)$

0

$f''(-1) = 24(-1)^2 = +$ $\cup$

$f''(1) = 24(1)^2 = +$ $\cup$

$$\boxed{\begin{array}{ll} \text{NO} & \text{IP} \\ \cup & (-\infty, \infty) \end{array}}$$

b) $f(x) = x\sqrt{x+1}$ D: $x+1 \geq 0$

$x \geq -1$

$f'(x) = (1)(x+1)^{1/2} + x\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}} = \frac{\sqrt{x+1}}{1} \cdot \frac{2\sqrt{x+1}}{2\sqrt{x+1}} + \frac{x}{2\sqrt{x+1}}$ LCD $= 2\sqrt{x+1}$

$$f'(x) = \frac{2(x+1)+x}{2\sqrt{x+1}} = \frac{2x+2+x}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

$$f''(x) = \frac{\dfrac{3(2)(x+1)^{1/2}}{(x+1)^{-1/2}} - \dfrac{(3x+2)(\cancel{2})(\cancel{\tfrac{1}{2}})(\cancel{x+1})^{-1/2}}{\cancel{(x+1)^{-1/2}}}}{(2\sqrt{x+1})^2}$$

GCF $= (x+1)^{-1/2}$

$$f''(x) = \frac{(x+1)^{-1/2}\left[6(x+1) - (3x+2)\right]}{4(x+1)^1} = \frac{6x+6-3x-2}{4(x+1)^{3/2}}$$

$$f''(x) = \frac{3x+4}{4(x+1)^{3/2}}$$

$f''(x) = 0$

$3x+4=0$

~~$x = -\frac{4}{3}$~~

$f''(x)$ DNE

$x+1=0$

$x=-1$

-1

$$f''(0) = \frac{3(0)+4}{4(0+1)^{3/2}} = \frac{+}{+} = +$$

$\cup$

D: $x \geq -1$

$$\boxed{\begin{array}{l}\text{NO IP} \\ \cup \ (-1, \infty)\end{array}}$$

c) $f(x)=\frac{1}{3}x^3-\frac{1}{2}x^2-2x+\frac{1}{3}$

$f'(x)=x^2-x-2$

$f''(x)=2x-1$

$\underline{f''(x)=0}$

$2x-1=0$

$x=\frac{1}{2}$

$f\left(\frac{1}{2}\right)=\frac{1}{3}\left(\frac{1}{2}\right)^3-\frac{1}{2}\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)+\frac{1}{3}$

$f\left(\frac{1}{2}\right)=\frac{1}{24}-\frac{1}{8}-1+\frac{1}{3}$

$f\left(\frac{1}{2}\right)=\frac{1}{24}-\frac{3}{24}-\frac{24}{24}+\frac{8}{24}$

$f\left(\frac{1}{2}\right)=\frac{-18}{24}=\frac{-3}{4}$

$(1/2, -3/4)$

$\frac{1}{2}$

$f''(0)=2(0)-1=-$

$f''(1)=2(1)-1=+$

$$\boxed{\begin{array}{l} \text{IP } (1/2, -3/4) \\ (-\infty, 1/2) \\ (1/2, \infty) \end{array}}$$

2. Find the relative extrema using the Second Derivative Test.

a) $f(x) = 3x^4 - 4x^3 - 12x^2$

$f'(x) = 12x^3 - 12x^2 - 24x$

$f'(x) = 0$

$12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$12x(x-2)(x+1) = 0$

$x = 0 \quad x = 2 \quad x = -1$

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 = 3 + 4 - 12 = -5$

$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 = 0$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 = 48 - 32 - 48 = -32$

$\boxed{\begin{array}{l}(-1,-5)\text{ min}\\(0,0)\text{ max}\\(2,-32)\text{ min}\end{array}}$

$f''(x) = 36x^2 - 24x - 24$

$f''(-1) = 36(-1)^2 - 24(-1) - 24 = 36 + 24 - 24 = + \cup$

$f''(0) = 36(0)^2 - 24(0) - 24 = -24 = \cap$

$f''(2) = 36(2)^2 - 24(2) - 24 = 144 - 48 - 24 = + \cup$

b) $f(x) = \sqrt{x^2+1}$ D: $\mathbb{R}$

$f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+1}}$

$f'(x) = 0$

$x = 0$

$f'(x)$ DNE

$x^2 + 1 = 0$

$\cancel{x = \pm i}$

$f(0) = \sqrt{0^2+1} = 1$

$\boxed{(0,1)\text{ min}}$

GCF = $(x^2+1)^{-1/2}$

$$f''(x) = \frac{\frac{(1)(x^2+1)^{\frac{1}{2}}}{(x^2+1)^{-\frac{1}{2}}} - \frac{(x)(\frac{1}{2})\cancel{(x^2+1)^{-\frac{1}{2}}}(\cancel{2}x)}{\cancel{(x^2+1)^{-\frac{1}{2}}}}}{\left(\sqrt{x^2+1}\right)^2}$$

$$f''(x) = \frac{(x^2+1)^{-1/2}\left[\cancel{x^2} + 1 - \cancel{x^2}\right]}{(x^2+1)^1}$$

$$f''(x) = \frac{1}{(x^2+1)^{3/2}}$$

$$f''(0) = \frac{1}{(0^2+1)^{3/2}} = \frac{+}{+} \quad \cup$$

3. A function f is continuous on $[-3,3]$ such that $f(-3)=4$ and $f(3)=1$. The functions f' and f'' have the properties given in the table below. Sketch the graph. $(-3,4)(3,1)$

	f'	f''
$-3<x<-1$	+	+
$x=-1$	*DNE*	*DNE*
$-1<x<1$	−	+
$x=1$	0	0
$1<x<3$	−	−