Integration by Substitution (U-Substitution)

Directions: Find the indefinite integral.

1. $\int x^4\left(7-x^5\right)^3 dx$ $= \int -\frac{1}{5} u^3 du = -\frac{1}{5}\int u^3 du = -\frac{1}{5}\frac{u^4}{4} = -\frac{1}{20}u^4$

$u = 7 - x^5$

$du = -5x^4 dx$

$-\frac{1}{5}du = x^4 dx$

$\boxed{= -\frac{1}{20}(7-x^5)^4 + C}$

2. $\int 8x\left(x^2-1\right)^{1/3} dx$ $= 8\int \frac{1}{2} u^{1/3} du = 4\int u^{1/3} du = 4 \cdot \frac{u^{4/3}}{4/3} = \cancel{4} \cdot \frac{3}{\cancel{4}} u^{4/3}$

$u = x^2 - 1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= 3u^{4/3} = \boxed{3(x^2-1)^{4/3} + C}$

3. $\int \frac{6x}{\sqrt{2x^2+1}}dx = 6\int \frac{1}{4}\frac{1}{\sqrt{u}}du = \frac{3}{2}\int u^{-1/2}du = \frac{3}{2}\frac{u^{1/2}}{1/2} = \frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{1}u^{1/2}$

$u = 2x^2+1$

$du = 4x\,dx$

$\frac{1}{4}du = x\,dx$

$= 3\sqrt{u} = \boxed{3\sqrt{2x^2+1}+C}$

4. $\int \sec^2\left(\frac{\theta}{4}\right)d\theta = \int 4\sec^2 u\,du = 4\int \sec^2 u\,du = 4\tan u$

$\boxed{= 4\tan\left(\frac{\theta}{4}\right)+C}$

$u = \frac{\theta}{4}$

$du = \frac{1}{4}d\theta$

$4\,du = d\theta$

5. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = \int 2\sin u\,du = 2\int \sin u\,du = -2\cos u$

$$\boxed{= -2\cos\sqrt{\theta} + C}$$

$u = \sqrt{\theta}$

$du = \frac{1}{2}\theta^{-1/2}\,d\theta$

$du = \frac{1}{2\sqrt{\theta}}\,d\theta$

$2\,du = \frac{1}{\sqrt{\theta}}\,d\theta$

6. $\int x\sqrt{3x+2}\,dx = \int \frac{1}{3}\left(\frac{u-2}{3}\right)\sqrt{u}\,du = \frac{1}{9}\int \sqrt{u}\,(u-2)\,du$

$u = 3x+2$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$x = \frac{u-2}{3}$

$= \frac{1}{9}\int u^{3/2} - 2u^{1/2}\,du$

$= \frac{1}{9}\left[\frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2}\right]$

$= \frac{1}{9}\left[\frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2}\right] = \frac{2}{45}u^{5/2} - \frac{4}{27}u^{3/2}$

$$\boxed{= \frac{2}{45}(3x+2)^{5/2} - \frac{4}{27}(3x+2)^{3/2} + C}$$

7. $\int x^2\sqrt{3-x}\,dx$ $= \int -(9-6u+u^2)\sqrt{u}\,du = -\int \sqrt{u}(9-6u+u^2)\,du$

$u = 3-x$

$du = -dx$

$-du = dx$

$x = 3-u$

$x^2 = 9-6u+u^2$

$= -\int 9u^{1/2} - 6u^{3/2} + u^{5/2}\,du$

$= -\left[9\cdot\frac{2}{3}u^{3/2} - 6\cdot\frac{2}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right] = -6u^{3/2} + \frac{12}{5}u^{5/2} - \frac{2}{7}u^{7/2}$

$\boxed{= -6(3-x)^{3/2} + \frac{12}{5}(3-x)^{5/2} - \frac{2}{7}(3-x)^{7/2} + C}$

Directions: Evaluate the definite integral.

8. $\int_0^2 (x^2+2x-4)^3(x+1)\,dx$ $= \int_0^2 \frac{1}{2}u^3\,du = \frac{1}{2}\int_0^2 u^3\,du = \frac{1}{2}\frac{u^4}{4} = \frac{1}{8}u^4$

$u = x^2+2x-4$

$du = 2x+2\,dx$

$du = 2(x+1)\,dx$

$\frac{1}{2}du = (x+1)\,dx$

$= \frac{1}{8}(x^2+2x-4)^4\Big]_0^2$

$= \frac{1}{8}\left[(2^2+2\cdot2-4)^4 - (0^2+2\cdot0-4)^4\right]$

$= \frac{1}{8}\left[4^4 - (-4)^4\right] = \frac{1}{8}\left[256-256\right]$

$= \frac{1}{8}(0) = \boxed{0}$

9. $\displaystyle\int_0^{\frac{\pi}{6}} \underline{(1-\cos 3x)}\,\underline{\sin 3x\,dx} = \int_0^{\pi/6} \frac{1}{3} u\,du = \frac{1}{3}\int_0^{\pi/6} u\,du = \frac{1}{3}\frac{u^2}{2} = \frac{1}{6}u^2$

$u = 1-\cos 3x$

$du = (\sin 3x)\cdot 3\,dx$

$\frac{1}{3}du = \sin 3x\,dx$

$= \frac{1}{6}(1-\cos 3x)^2\Big]_0^{\pi/6}$

$= \frac{1}{6}\left[(1-\cos \pi/2)^2 - (1-\cos 0)^2\right] = \frac{1}{6}\left[1^2 - 0^2\right] = \boxed{\frac{1}{6}}$