Areas of Surfaces of Revolution

The area of a surface $f(x)$ rotated about the x-axis:

$$S = \int_a^b 2\pi \bullet \text{radius} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

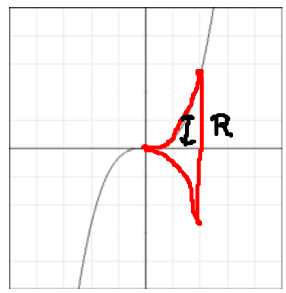
The area of a surface $g(y)$ rotated about the y-axis:

$$S = \int_c^d 2\pi \bullet \text{radius} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Directions: For questions 1 through 3, find the area of the surface obtained by rotating the curve about the x-axis.

1. $y = \frac{1}{3}x^3,\ 0 \le x \le 2$

Radius $= y = \frac{1}{3}x^3$



$y' = x^2$

$(y')^2 = x^4$

$$2\pi \int_0^2 \underbrace{\frac{1}{3}x^3}_{R} \sqrt{1+x^4}\, dx = \frac{2\pi}{3}\int_0^2 x^3 \sqrt{1+x^4}\, dx$$

$u = 1+x^4$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$$\frac{\cancel{2}^{1}\pi}{3}\cdot\frac{1}{\cancel{4}_{2}}\int_0^2 \sqrt{u}\,du = \frac{\pi}{6}\int_0^2 u^{1/2} = \frac{\pi}{\cancel{6}_{3}}\cdot\frac{\cancel{2}^{1}}{3}u^{3/2} = \frac{\pi}{9}(1+x^4)^{3/2}\Big]_0^2$$

$$\frac{\pi}{9}\left[(1+2^4)^{3/2} - (1+0^4)^{3/2}\right] = \frac{\pi}{9}\left[17^{3/2} - 1^{3/2}\right] = \boxed{\frac{\pi}{9}\left(17\sqrt{17}-1\right) \text{ OR } 24.1179}$$

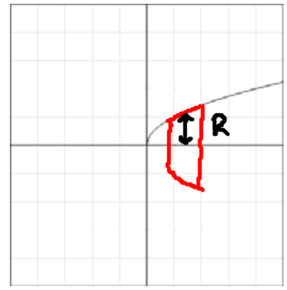
$(\sqrt{17})^3$

$\sqrt{17}\cdot\sqrt{17}\cdot\sqrt{17}$

$17\sqrt{17}$

2. $y=\sqrt{x}$, $\left[\frac{3}{4}, 2\right]$

Radius $= y = \sqrt{x}$



$y = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$(y')^2 = \left(\frac{1}{2\sqrt{x}}\right)^2 = \frac{1}{4x}$

$$2\pi\int_{3/4}^{2} \underbrace{\sqrt{x}}_{R}\sqrt{1+\frac{1}{4x}}\,dx = 2\pi\int_{3/4}^{2}\sqrt{x\left(1+\frac{1}{4x}\right)}\,dx = 2\pi\int_{3/4}^{2}\sqrt{x+\frac{1}{4}}\,dx$$

$u = x + \frac{1}{4}$

$du = dx$

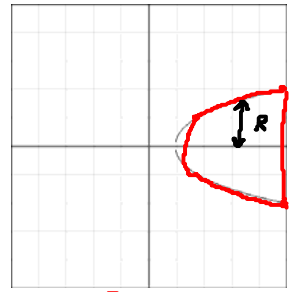
$$2\pi\int_{3/4}^{2}\sqrt{u}\,du = 2\pi\int_{3/4}^{2}u^{1/2}\,du = 2\pi\cdot\frac{2}{3}u^{3/2} = \frac{4\pi}{3}\left(x+\frac{1}{4}\right)^{3/2}\Big]_{3/4}^{2}$$

$$\frac{4\pi}{3}\left[\left(2+\frac{1}{4}\right)^{3/2} - \left(\frac{3}{4}+\frac{1}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left[\left(\frac{9}{4}\right)^{3/2} - 1^{3/2}\right]$$

$$\frac{4\pi}{3}\left(\frac{27}{8}-1\right) = \frac{4\pi}{3}\left(\frac{27-8}{8}\right) = \frac{\cancel{4}^{1}\pi}{3}\left(\frac{19}{\cancel{8}_{2}}\right) = \boxed{\frac{19\pi}{6} \text{ OR } 9.948}$$

3. $x=\frac{1}{3}\left(y^2+2\right)^{\frac{3}{2}}$, $1\le y\le 2$

Radius = y



$$x' = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(y^2+2)^{1/2}\cancel{2}y = y\sqrt{y^2+2}$$

$$(x')^2 = \left(y\sqrt{y^2+2}\right)^2 = y^2(y^2+2) = y^4+2y^2$$

$$2\pi\int_1^2 \underbrace{y}_{R}\sqrt{1+y^4+2y^2}\,dy = 2\pi\int_1^2 y\sqrt{y^4+2y^2+1}\,dy = 2\pi\int_1^2 y\sqrt{(y^2+1)^2}\,dy$$

$$2\pi\int_1^2 y(y^2+1)\,dy = 2\pi\int_1^2 y^3+y\,dy = 2\pi\left[\frac{y^4}{4}+\frac{y^2}{2}\right]_1^2$$

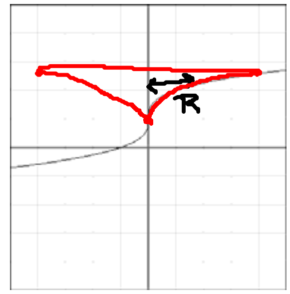
$$2\pi\left[\left(\frac{2^4}{4}+\frac{2^2}{2}\right)-\left(\frac{1^4}{4}+\frac{1^2}{2}\right)\right] = 2\pi\left[\frac{16}{4}+\frac{4}{2}-\frac{1}{4}-\frac{1}{2}\right]$$

$$2\pi\left[4+2-\frac{1}{4}-\frac{1}{2}\right] = 2\pi\left[6-\frac{1}{4}-\frac{1}{2}\right] = 2\pi\left[\frac{24-1-2}{4}\right]$$

$$\overset{1}{\cancel{2}}\pi\left[\frac{21}{\cancel{4}_2}\right] = \boxed{\frac{21\pi}{2} \text{ OR } 32.9867}$$

Directions: For questions 4 and 5, find the area of the surface obtained by rotating the curve about the y-axis.

4. $y=\sqrt[3]{x}+1,\ [0,4]$



$y = x^{1/3}+1$

Radius = x

$y' = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$(y')^2 = \left(\frac{1}{3x^{2/3}}\right)^2 = \frac{1}{9x^{4/3}}$

$$2\pi\int_0^4 \underbrace{x}_{R}\sqrt{1+\frac{1}{9x^{4/3}}}\,dx = 2\pi\int_0^4 x\sqrt{\frac{9x^{4/3}+1}{9x^{4/3}}}\,dx$$

$$2\pi\int_0^4 \frac{x}{3x^{2/3}}\sqrt{9x^{4/3}+1}\,dx = \frac{2\pi}{3}\int_0^4 x^{1/3}\sqrt{9x^{4/3}+1}\,dx$$

$u = 9x^{4/3}+1$

$du = \frac{36}{3}x^{1/3}dx$

$du = 12x^{1/3}dx$

$\frac{1}{12}du = x^{1/3}dx$

$$\frac{\cancelto{1}{2}\pi}{3}\cdot\frac{1}{\cancelto{6}{12}}\int_0^4\sqrt{u}\,du = \frac{\pi}{18}\int_0^4 u^{1/2}du = \frac{\pi}{\cancelto{9}{18}}\cdot\frac{\cancelto{1}{2}}{3}u^{3/2} = \frac{\pi}{27}\left(9x^{4/3}+1\right)^{3/2}\Big]_0^4$$

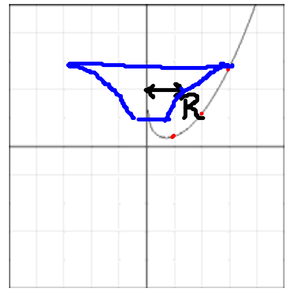
$$\frac{\pi}{27}\left[\left(9\cdot 4^{4/3}+1\right)^{3/2} - \left(9\cdot 0^{4/3}+1\right)^{3/2}\right] = \frac{\pi}{27}\left[\left(9\cdot 4\sqrt[3]{4}+1\right)^{3/2} - 1^{3/2}\right]$$

$$4^{4/3} = \sqrt[3]{256} = \sqrt[3]{64\cdot 4} = 4\sqrt[3]{4}$$

$$\boxed{\frac{\pi}{27}\left[\left(36\sqrt[3]{4}+1\right)^{3/2}-1\right] \text{ OR } 51.4743}$$

5. $x = \dfrac{y^2 - \ln y}{2\sqrt{2}}$, $1 \le y \le 3$

$y = \dfrac{x^2 - \ln x}{2\sqrt{2}}$



$(1, 1/2)$ $(1/2, 1)$

$(2, 1.1)$ $(1.1, 2)$

$(3, 2.8)$ $(2.8, 3)$

$x = \frac{1}{2\sqrt{2}}(y^2 - \ln y)$

$x' = \frac{1}{2\sqrt{2}}\left(2y - \frac{1}{y}\right) = \frac{1}{2\sqrt{2}}\left(\frac{2y^2-1}{y}\right)$

$(x')^2 = \left(\frac{2y^2-1}{2\sqrt{2}\,y}\right)^2 = \frac{4y^4 - 4y^2 + 1}{8y^2}$

Radius $= x = \dfrac{y^2 - \ln y}{2\sqrt{2}}$

$\cancel{2}\pi \int_1^3 \underbrace{\frac{y^2 - \ln y}{\cancel{2}\sqrt{2}}}_{\text{Radius}} \sqrt{1 + \frac{4y^4 - 4y^2 + 1}{8y^2}}\, dy$

$\frac{\pi}{\sqrt{2}} \int_1^3 (y^2 - \ln y) \sqrt{\frac{8y^2 + 4y^4 - 4y^2 + 1}{8y^2}}\, dy = \frac{\pi}{\sqrt{2}} \int_1 (y^2 - \ln y) \sqrt{\frac{4y^4 + 4y^2 + 1}{8y^2}}\, dy$

$\frac{\pi}{\sqrt{2}} \int_1^3 (y^2 - \ln y) \sqrt{\frac{(2y^2+1)^2}{8y^2}}\, dy = \frac{\pi}{\sqrt{2}} \int_1^3 (y^2 - \ln y)\frac{(2y^2+1)}{2\sqrt{2}\,y}\, dy$

$\frac{\pi}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} \int_1^3 \frac{(y^2 - \ln y)(2y^2+1)}{y}\, dy = \frac{\pi}{4} \int_1^3 \frac{2y^4 + y^2 - 2y^2 \ln y - \ln y}{y}\, dy$

$\frac{\pi}{4} \int_1^3 \frac{2y^4}{y} + \frac{y^2}{y} - \frac{2y^2 \ln y}{y} - \frac{\ln y}{y}\, dy = \frac{\pi}{4} \int_1 \underbrace{2y^3 + y}_{} - \underbrace{2y \ln y}_{} - \underbrace{\frac{\ln y}{y}}_{} dy$

$\frac{\pi}{4}\left[\int_1^3 2y^3 + y\, dy - 2 \underbrace{\int_1^3 y \ln y\, dy}_{\text{Integration by Parts}} - \underbrace{\int_1^3 \frac{\ln y}{y}\, dy}_{\text{u-substitution}} \right.$

$u = \ln y$ $\quad dv = y\,dy$

$du = \frac{1}{y}dy$ $\quad v = \frac{y^2}{2}$

$u = \ln y$

$du = \frac{1}{y}dy$

$\frac{\pi}{4}\left[\frac{2y^4}{4} + \frac{y^2}{2} - 2\left(\frac{y^2 \ln y}{2} - \frac{1}{2}\int \frac{y^2}{y}\, dy\right) - \int u\, du\right]$

$\frac{\pi}{4}\left[\frac{y^4}{2} + \frac{y^2}{2} - y^2 \ln y + \int y\, dy - \frac{u^2}{2}\right]$

$\frac{\pi}{4}\left[\frac{y^4}{2} + \frac{y^2}{2} - y^2 \ln y + \frac{y^2}{2} - \frac{(\ln y)^2}{2}\right]_1^3$

$\frac{\pi}{4}\left[\left(\frac{81}{2} + \frac{9}{2} - 9\ln 3 + \frac{9}{2} - \frac{(\ln 3)^2}{2}\right) - \left(\frac{1}{2} + \frac{1}{2} - \ln 1 + \frac{1}{2} - \frac{(\ln 1)^2}{2}\right)\right]$

$$\frac{\pi}{4}\left[\frac{99}{2} - 9\ln 3 - \frac{(\ln 3)^2}{2} - \frac{3}{2}\right] = \frac{\pi}{4}\left[\frac{96}{2} - \frac{18\ln 3}{2} - \frac{(\ln 3)^2}{2}\right]$$

$$\boxed{\frac{\pi}{8}\left[96 - 18\ln 3 - (\ln 3)^2\right] \text{ OR } 29.4595}$$