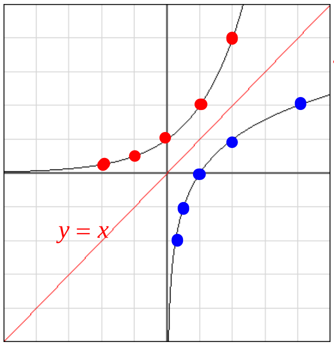
Inverse Functions

$f(x) = 2^x$

D: $\mathbb{R}$
R: $y > 0$



y-int: (0,1)
HA: y=0

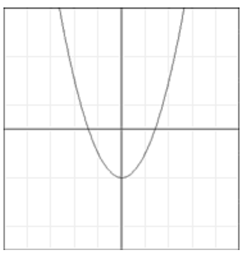
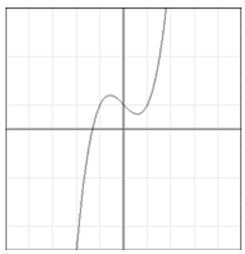
$f^{-1}(x) = \log_2 x$
D: $x > 0$
R: $\mathbb{R}$
x-int: (1,0)
VA: x=0

$f(x) = 2^x$

x	2^x	y		
-2	2^{-2}	$1/4$	$(-2, 1/4)$	$(1/4, -2)$
-1	2^{-1}	$1/2$	$(-1, 1/2)$	$(1/2, -1)$
0	2^0	1	$(0,1)$	$(1,0)$
1	2^1	2	$(1,2)$	$(2,1)$
2	2^2	4	$(2,4)$	$(4,2)$

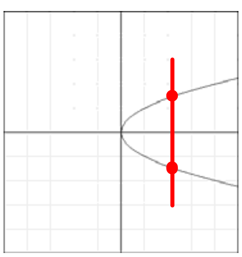
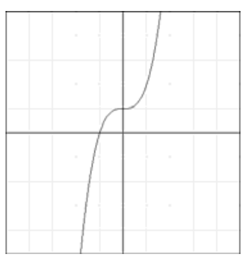
$y = 2^x$
$x = 2^y$
$\log_2 x = \log_2 2^y$
$\log_2 x = y$
$y = \log_2 x$
$f^{-1}(x) = \log_2 x$

Vertical Line Test - Determines if the graph is a function.

✓ ✓

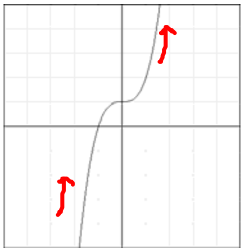
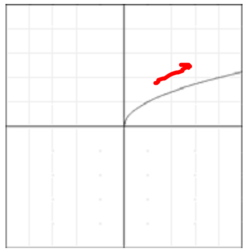
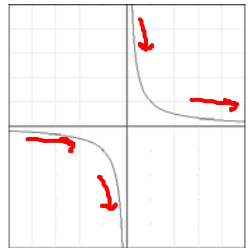
Horizontal Line Test - Determines if the function is one-to-one.

X ✓

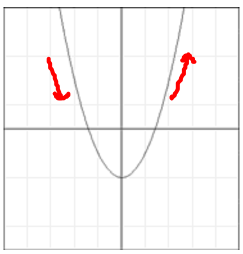
not a function

One-to-One Function - A graph that passes both the vertical and horizontal line tests. The graph is a function and it has an inverse.

✓ ✓ ✓

Monotonic Function - A function that is always increasing or always decreasing.



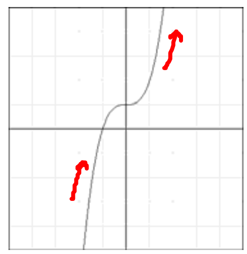
$f(x)=x^2-2$

$f'(x)=2x$

$f'(-1)=-2$

$f'(1)=2$

not monotonic



$f(x)=x^3+1$

$f'(x)=3x^2$

$f'(x)>0$

monotonic

If $f(x)$ and $g(x)$ are inverse functions, then $f(g(x))=g(f(x))=x$.

$f(x)=x^3$

$g(x)=\sqrt[3]{x}$

$f(g(x)) = (\sqrt[3]{x})^3 = x$

$g(f(x)) = \sqrt[3]{x^3} = x$

Directions: For questions 1 through 7, determine if the function has an inverse.

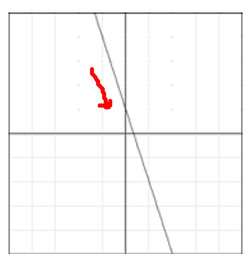
1. $f(x)=-3x+1$

D: $\mathbb{R}$

$f'(x)=-3$

$f'(x)<0$

Decreasing
monotonic
Has an inverse



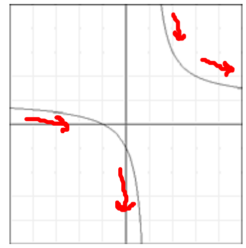
2. $f(x)=\frac{x+1}{x-1}$

D: $x \neq 1$

$f'(x) = \frac{1(x-1)-1(x+1)}{(x-1)^2}$

$f'(x) = \frac{x-1-x-1}{(x-1)^2}$

$f'(x) = \frac{-2}{(x-1)^2}$



$f'(x) < 0$

$\boxed{\text{Decreasing, monotonic, has an inverse}}$

3. $f(x)=\sqrt{2-x}$

$2-x \geq 0$

$2 \geq x$

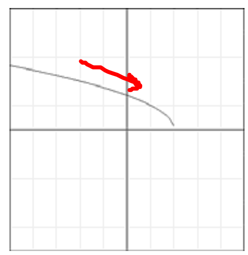
$x \leq 2$

D: $x \leq 2$

$f(x) = (2-x)^{1/2}$

$f'(x) = \frac{1}{2}(2-x)^{-1/2}(-1)$

$f'(x) = \frac{-1}{2\sqrt{2-x}}$



$f'(x) < 0$

$\boxed{\text{Decreasing, monotonic, has an inverse}}$

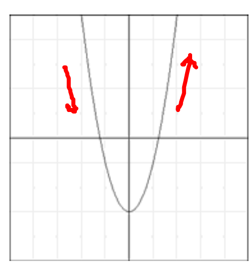
4. $f(x)=2x^2-3$

D: $\mathbb{R}$

$f'(x) = 4x$

$f'(-1) = -4$ ↓

$f'(1) = 4$ ↑



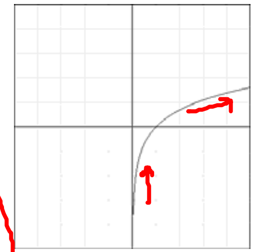
$\boxed{\text{Not monotonic Does not have an inverse}}$

5. $f(x) = \ln x$

$D: x > 0$

$f'(x) = \frac{1}{x}$

$f'(x) > 0, \quad x > 0$



Increasing
Monotonic
Has an inverse

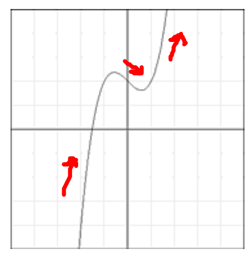
6. $f(x) = x^3 - x + 2$

$D: \mathbb{R}$

$f'(x) = 3x^2 - 1$

$f'(0) = -1 \downarrow$

$f'(2) = 11 \uparrow$

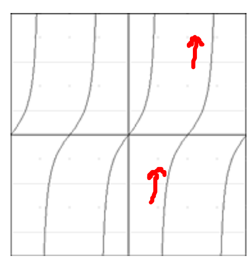


Not monotonic
Does not have an inverse

7. $f(x) = \tan x$

Asymptotes: $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots \frac{(2n-1)\pi}{2}$

$D: x \neq \frac{(2n-1)\pi}{2}$



$f'(x) = \sec^2 x$

$f'(x) > 0$

Increasing, monotonic, has an inverse $\left(\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}\right)$

Directions: For questions 8 through 12, find the inverse of each function.

8. $f(x) = -3x + 1$

$y = -3x + 1$

$x = -3y + 1$

$x - 1 = -3y$

$y = \frac{x-1}{-3}$

$y = -\frac{1}{3}x + \frac{1}{3}$

$\boxed{f^{-1}(x) = -\frac{1}{3}x + \frac{1}{3}}$

9. $f(x) = \frac{3x+1}{2x-1}$

$y = \frac{3x+1}{2x-1}$

$\frac{x}{1} = \frac{3y+1}{2y-1}$

$x(2y-1) = 3y + 1$

$2xy - x = 3y + 1$ → OR

$2xy - 3y = x + 1$

$y(2x-3) = x + 1$

$y = \frac{x+1}{2x-3}$

$\boxed{f^{-1}(x) = \frac{x+1}{2x+3}}$

$2xy - x = 3y + 1$

$-x - 1 = 3y - 2xy$

$-x - 1 = y(3 - 2x)$

$y = \frac{-x-1}{3-2x}$

$\boxed{f^{-1}(x) = \frac{-x-1}{3-2x}}$

10. $f(x)=\sqrt{2-x}$

$2-x\ge 0$

$2\ge x$

D: $x\le 2$

R: $y\ge 0$

$y=\sqrt{2-x}$

$(x)^2=(\sqrt{2-y})^2$

$x^2=2-y$

$y+x^2=2$

$y=-x^2+2$

$\boxed{f^{-1}(x)=-x^2+2,\quad x\ge 0}$

11. $f(x)=2x^2-3,\ x\ge 0$

$y=2x^2-3$

$x=2y^2-3$

$2y^2=x+3$

$\sqrt{y^2}=\sqrt{\frac{x+3}{2}}$

$y=\pm\sqrt{\frac{x+3}{2}}$

$\boxed{f^{-1}(x)=\sqrt{\frac{x+3}{2}}}$

12. $f(x) = \ln x$

$y = \ln x$

$x = \ln y$

$e^x = y$

$y = e^x$

$\boxed{f^{-1}(x) = e^x}$

Directions: For question 13, show that the functions are inverses of each other.

13. $f(x) = \dfrac{1}{x-2} + 1$

$g(x) = \dfrac{2x-1}{x-1}$

$f(g(x)) = \dfrac{1}{\left(\frac{2x-1}{x-1}\right) - \frac{2\cdot(x-1)}{1\cdot(x-1)}} + 1 = \dfrac{1}{\frac{2x-1-2(x-1)}{x-1}} + 1 = \dfrac{1}{\frac{2x-1-2x+2}{x-1}} + 1$

$= \dfrac{1}{\frac{1}{x-1}} + 1 = \dfrac{x-1}{1} + 1 = x - 1 + 1 = \boxed{x}$

$g(f(x)) = \dfrac{2\left(\frac{1}{x-2} + 1\right) - 1}{\left(\frac{1}{x-2} + 1\right) - 1} = \dfrac{\frac{2}{x-2} + 2 - 1}{\frac{1}{x-2} + 1 - 1} = \dfrac{\frac{2}{x-2} + \frac{1\cdot(x-2)}{1\cdot(x-2)}}{\frac{1}{x-2}}$

$= \dfrac{\frac{2+1(x-2)}{x-2}}{\frac{1}{x-2}} = \dfrac{2+x-2}{1} = \boxed{x}$

Derivative of the Inverse Function: $\left(f^{-1}\right)'(x) = \dfrac{1}{f'(g(x))}$

Steps to find the derivative of the inverse at $x = a$.

1. Set $f(x)$ equal to a and solve for x.
2. Find $f'(x)$ at the value of x found in step 1.
3. Take the recriprocal of the value found in step 2.

Directions: For questions 14 and 15, find $\left(f^{-1}\right)'(a)$.

14. $f(x) = \dfrac{\ln e^{3x}}{x-1}$, $a = 2$

$f(x) = \dfrac{3x}{x-1}$

Step1: $f(x) = 2$

$\dfrac{3x}{x-1} = \dfrac{2}{1}$

$3x = 2(x-1)$

$3x = 2x - 2$

$x = -2$

Step2: $f'(x) = \dfrac{3(x-1) - 1(3x)}{(x-1)^2}$

$f'(x) = \dfrac{3x - 3 - 3x}{(x-1)^2}$

$f'(x) = \dfrac{-3}{(x-1)^2}$

$f'(-2) = \dfrac{-3}{(-2-1)^2}$

$f'(-2) = \dfrac{-3}{(-3)^2}$

$f'(-2) = -\dfrac{3}{9}$

$f'(-2) = -\dfrac{1}{3}$

Step3: $\boxed{\left(f^{-1}\right)'(2) = -3}$

15. $f(x) = x^3 - x^2 + 2x + 2$, $a = 4$

Step1: $f(x) = 4$

$x^3 - x^2 + 2x + 2 = 4$

$x^3 - x^2 + 2x - 2 = 0$

Factor by Grouping

$x^2(x-1) + 2(x-1) = 0$

$(x^2 + 2)(x-1) = 0$

$x^2 + 2 = 0$ $\quad$ $x - 1 = 0$

$\sqrt{x^2} = \sqrt{-2}$ $\quad$ $x = 1$

$x = \pm\sqrt{-2}$

$x = \pm i\sqrt{2}$

Step2: $f'(x) = 3x^2 - 2x + 2$

$f'(1) = 3(1)^2 - 2(1) + 2$

$f'(1) = 3$

Step3: $\boxed{\left(f^{-1}\right)'(4) = \dfrac{1}{3}}$