Natural Logarithms - Differentiation and Integration

Derivative of the Natural Logarithmic Function

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad x > 0$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\bullet\frac{du}{dx} \qquad u > 0$$

1. Find the derivative of each.

a) $\ln 2x$

$$\frac{d}{dx}\ln 2x = \frac{1}{2x}\cdot 2 = \frac{\cancel{2}}{\cancel{2}x} = \boxed{\frac{1}{x}}$$

b) $\ln\left(x^2+3\right)$

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \boxed{\frac{2x}{x^2+3}}$$

c) $\ln(\ln x)$

$$\frac{d}{dx}\ln\underbrace{(\ln x)} = \frac{1}{\ln x}\cdot\frac{1}{x} = \boxed{\frac{1}{x\ln x}}$$

d) $\ln \frac{1}{x\sqrt{x+1}} = \ln 1 - \ln x\sqrt{x+1}$

$= \ln 1 - (\ln x + \ln \sqrt{x+1})$

$= \ln 1 - \ln x - \ln \sqrt{x+1}$

$\frac{d}{dx} \ln 1 - \frac{d}{dx} \ln x - \frac{d}{dx} \ln \sqrt{x+1}$

$\frac{1}{1} \cdot 0 - \frac{1}{x} \cdot 1 - \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2\sqrt{x+1}}$

$0 - \frac{1}{x} - \frac{1}{2(x+1)} = \boxed{-\frac{1}{x} - \frac{1}{2(x+1)}}$

$(x+1)^{\frac{1}{2}}$

$\frac{d}{dx}(x+1)^{1/2} = \frac{1}{2}(x+1)^{-1/2}(1)$

$= \frac{1}{2\sqrt{x+1}}$

e) $y = \frac{(x^2+1)\sqrt{x+3}}{x-1}$ ln

$\ln y = \ln \frac{(x^2+1)(\sqrt{x+3})}{x-1}$

$\ln y = \ln(x^2+1) + \ln\sqrt{x+3} - \ln(x-1)$

$\frac{1}{y} \cdot y' = \frac{1}{x^2+1} \cdot 2x + \frac{1}{\sqrt{x+3}} \cdot \frac{1}{2\sqrt{x+3}} - \frac{1}{x-1} \cdot 1$

$\cancel{y} \cdot \frac{1}{\cancel{y}} \cdot y' = \left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right) y$

$$\boxed{y' = \left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right) \cdot \frac{(x^2+1)\sqrt{x+3}}{x-1}}$$

$\sqrt{x+3} = (x+3)^{1/2}$

$\frac{d}{dx}(x+3)^{1/2}$

$= \frac{1}{2}(x+3)^{-1/2}(1)$

$= \frac{1}{2\sqrt{x+3}}$

Integrating the Natural Logarithmic Function

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

2. Integrate.

a) $\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx = 2\ln|x| = \boxed{\ln x^2 + C}$

b) $\int \frac{1}{4x-1}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| = \frac{1}{4}\ln|4x-1|$

$u = 4x-1$

$\frac{du}{4} = \frac{4\,dx}{4}$

$\frac{1}{4}du = dx$

$= \ln\left(|4x-1|\right)^{1/4} = \boxed{\ln\sqrt[4]{|4x-1|} + C}$

c) $\displaystyle\int_0^2 \frac{2x}{x^2-5}\,dx = \int_0^2 \frac{1}{u}\,du = \ln|u|\Big|_0^2 = \ln|x^2-5|\Big|_0^2$

$u = x^2-5$

$du = 2x\,dx \qquad = \ln|2^2-5| - \ln|0^2-5|$

$$= \ln|-1| - \ln|-5|$$

$$= \underbrace{\ln 1}_{\substack{\parallel \\ 0}} - \ln 5$$

$$\boxed{= -\ln 5 = -1.6094}$$

d) $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx = \int 1 + \frac{x}{x^2+1}\,dx = \int 1\,dx + \int \frac{x}{x^2+1}\,dx$

$$\begin{array}{r|l} & 1 + \frac{x}{x^2+1} \\ \hline x^2+1 & x^2 + x + 1 \\ & \cancel{-x^2} \qquad \cancel{-1} \\ \hline & x \end{array}$$

$\frac{x^2}{x^2} = 1$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = \frac{2x\,dx}{2}$

$\frac{1}{2}du = x\,dx$

$$\int 1\,dx + \frac{1}{2}\int \frac{1}{u}\,du$$

$$x + \frac{1}{2}\ln|u| = x + \frac{1}{2}\ln(x^2+1) = \boxed{x + \ln\sqrt{x^2+1} + C}$$

e) $\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta$ $= \frac{1}{2}\cdot 4 \int_{-\pi/2}^{\pi/2} \frac{1}{u}\,du = 2\int_{-\pi/2}^{\pi/2} \frac{1}{u}\,du$

$u = 3+2\sin\theta$

$\frac{du}{2} = \frac{2\cos\theta\,d\theta}{2}$

$\frac{1}{2}du = \cos\theta\,d\theta$ $= 2\ln|u|\Big|_{-\pi/2}^{\pi/2} = 2\ln|3+2\sin\theta|\Big|_{-\pi/2}^{\pi/2}$

$2\left[\ln|\underset{3+2(1)}{3+2\sin\pi/2}| - \ln|\underset{3+2(-1)}{3+2\sin{-\pi/2}}|\right]$

$2\left[\ln|5| - \ln|1|\right] = 2\left[\ln 5 - \underbrace{\ln 1}_{=0}\right] = \boxed{2\ln 5 = 3.2189}$