Exponential Functions - Differentiation and Integration

Differentiation

$$\frac{d}{dx}\left(e^x\right)=e^x$$

$$\frac{d}{dx}\left(e^u\right)=e^u\frac{du}{dx}$$

$y=e^x$

$\ln e^x = x$

$\frac{d}{dx}(\ln e^x) = \frac{d}{dx}(x)$

$\cancel{e^x}\cdot\frac{1}{\cancel{e^x}}\cdot\frac{d}{dx}(e^x) = 1\cdot e^x$

$\boxed{\frac{d}{dx}(e^x)=e^x}$

Integration

$$\int e^x dx = e^x + C$$

Directions: For questions 1 through 3, find the derivative.

1. $y=e^{3x}$

$y' = e^{3x}\cdot 3$

$\boxed{y' = 3e^{3x}}$

2. $y=e^{\sqrt{x}}$

$y' = e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$

$\boxed{y' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}}$

3. $y=\ln\left(\frac{1+e^x}{1-e^x}\right)$

$$y=\ln(1+e^x)-\ln(1-e^x)$$

$$y'=\frac{1}{1+e^x}\cdot e^x-\frac{1}{1-e^x}\cdot(-e^x)$$

$$y'=\frac{e^x}{1+e^x}\cdot(1-e^x)+\frac{e^x}{1-e^x}\cdot(1+e^x)$$

$$LCD=(1+e^x)(1-e^x)$$

$$y'=\frac{e^x(1-e^x)+e^x(1+e^x)}{(1+e^x)(1-e^x)}$$

$$y'=\frac{e^x(1-e^x+1+e^x)}{(1+e^x)(1-e^x)}$$

$$\boxed{y'=\frac{2e^x}{(1+e^x)(1-e^x)}}$$

Directions: For question 4, use implicit differentiation to find $\frac{dy}{dx}$.

4. $e^{xy}-\ln(xy)+xe^y=e^x\ln x^3$

$$e^{xy}\cdot\left(1y+x\frac{dy}{dx}\right)-\frac{1}{xy}\cdot\left(1y+x\frac{dy}{dx}\right)+1\cdot e^y+xe^y\frac{dy}{dx}=e^x\ln x^3+e^x\cdot\frac{1}{x^3}\cdot 3x^2$$

$$ye^{xy}+xe^{xy}\frac{dy}{dx}-\frac{1}{x}-\frac{1}{y}\frac{dy}{dx}+e^y+xe^y\frac{dy}{dx}=e^x\ln x^3+\frac{3e^x}{x}$$

$$\frac{dy}{dx}\left[xe^{xy}-\frac{1}{y}+xe^y\right]=e^x\ln x^3+\frac{3e^x}{x}-ye^{xy}+\frac{1}{x}-e^y$$

$$LCD=y \qquad LCD=x$$

$$\frac{dy}{dx}\left[\frac{xye^{xy}-1+xye^y}{y}\right]=\frac{xe^x\ln x^3+3e^x-xye^{xy}+1-xe^y}{x}$$

$$\frac{dy}{dx}=\frac{xe^x\ln x^3+3e^x-xye^{xy}+1-xe^y}{x}\cdot\frac{y}{xye^{xy}-1+xye^y}$$

$$\boxed{\frac{dy}{dx}=\frac{y(xe^x\ln x^3+3e^x-xye^{xy}+1-xe^y)}{x(xye^{xy}-1+xye^y)}}$$

Directions: For questions 5 through 9, evaluate the definite integral.

5. $\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

$u = e^x + e^{-x}$

$du = e^x + e^{-x}(-1)\,dx$

$du = e^x - e^{-x}\,dx$

$\int_0^1 \frac{1}{u}\,du = \ln|u| = \ln(e^x + e^{-x})\Big]_0^1 = \ln(e^1 + e^{-1}) - \ln(e^0 + e^0)$

$= \ln\left(e + \frac{1}{e}\right) - \ln 2 \quad = \ln\left(\frac{e^2+1}{e}\right) - \ln 2$

$= \ln(e^2+1) - \ln e - \ln 2 = \boxed{\ln\left(\frac{e^2+1}{2}\right) - 1}$

6. $\int_0^{\sqrt{2}} \frac{e^{2x} - 3e^x + 2}{e^x}\,dx = \int_0^{\sqrt{2}} \frac{e^{2x}}{e^x} - \frac{3e^x}{e^x} + \frac{2}{e^x}\,dx = \int_0^{\sqrt{2}} e^x - 3 + 2e^{-x}\,dx$

$\int_0^{\sqrt{2}} e^x - 3\,dx + 2\int_0^{\sqrt{2}} e^{-x}\,dx = \int_0^{\sqrt{2}} e^x - 3\,dx - 2\int_0^{\sqrt{2}} e^u\,du$

$u = -x$

$du = -dx$

$-du = dx$

$e^x - 3x - 2e^u = e^x - 3x - 2e^{-x}\Big]_0^{\sqrt{2}}$

$(e^{\sqrt{2}} - 3\sqrt{2} - 2e^{-\sqrt{2}}) - (e^0 - 3\cdot 0 - 2e^0)$

$e^{\sqrt{2}} - 3\sqrt{2} - 2e^{-\sqrt{2}} - 1 + 2$

$\boxed{e^{\sqrt{2}} - 3\sqrt{2} - 2e^{-\sqrt{2}} + 1}$

7. $\int_{\sqrt{2}}^{\sqrt{3}} x^2 e^{x^3}\,dx$

$u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$= \frac{1}{3}\int_{\sqrt{2}}^{\sqrt{3}} e^u\,du = \frac{1}{3}e^u = \frac{1}{3}e^{x^3}\Big]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \frac{1}{3}\left[e^{(\sqrt{3})^3} - e^{(\sqrt{2})^3}\right] = \frac{1}{3}\left(e^{3\sqrt{3}} - e^{2\sqrt{2}}\right)$$

$$= \boxed{\frac{e^{3\sqrt{3}} - e^{2\sqrt{2}}}{3}}$$

8. $\int_{\ln 2}^{\ln 3} \frac{e^{\frac{1}{x}}}{x^2}\,dx$

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}\,dx$

$-du = \frac{1}{x^2}\,dx$

$$= -\int_{\ln 2}^{\ln 3} e^u\,du = -e^u = -e^{\frac{1}{x}}\Big]_{\ln 2}^{\ln 3}$$

$$= -\left[e^{\frac{1}{\ln 3}} - e^{\frac{1}{\ln 2}}\right] = -e^{\frac{1}{\ln 3}} + e^{\frac{1}{\ln 2}}$$

$$= \boxed{e^{\frac{1}{\ln 2}} - e^{\frac{1}{\ln 3}}}$$

9. $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{\cos x \cdot e^{\csc x}}{1-\cos^2 x}\, dx$

$u = \csc x$

$du = -\csc x \cot x\, dx$

$-du = \csc x \cot x\, dx$

$$\frac{\cos x}{1-\cos^2 x} = \frac{\cos x}{\sin^2 x} = \frac{\cos x}{\sin x \cdot \sin x}$$

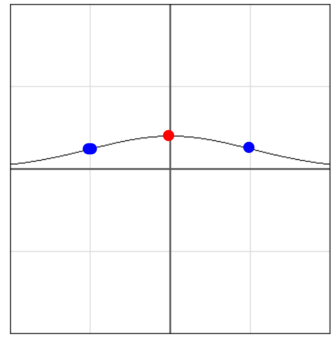
$$\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = \cot x \cdot \csc x$$

$$= -\int_{\pi/2}^{3\pi/4} e^u\, du = -e^u = -e^{\csc x}\Big]_{\pi/2}^{3\pi/4} = -\left[e^{\csc\frac{3\pi}{4}} - e^{\csc\frac{\pi}{2}}\right]$$

$$= -\left[e^{\sqrt{2}} - e^1\right] = -e^{\sqrt{2}} + e = \boxed{e - e^{\sqrt{2}}}$$

Directions: For question 10, find the extrema and points of inflection.

10. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



$$f'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot \frac{-2x}{2}$$

$$f'(x) = \frac{-x}{\sqrt{2\pi}\, e^{\frac{x^2}{2}}}$$

$-x = 0$

$x = 0$

$f(0) = \frac{1}{\sqrt{2\pi}}$

$$\boxed{\text{Extrema } \left(0, \frac{1}{\sqrt{2\pi}}\right)}$$

$$f'(x) = \frac{1}{\sqrt{2\pi}} \left(-x \cdot e^{-x^2/2}\right)$$

$$f''(x) = \frac{1}{\sqrt{2\pi}} \left[-1 \cdot e^{-\frac{x^2}{2}} + -x \cdot e^{-\frac{x^2}{2}} \cdot \frac{-2x}{2}\right]$$

$$f''(x) = \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{e^{x^2/2}} + \frac{x^2}{e^{x^2/2}}\right]$$

$$f''(x) = \frac{x^2 - 1}{\sqrt{2\pi} \cdot e^{x^2/2}}$$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

$$f(\pm 1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-1/2} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{2\pi e}}$$

$$\boxed{\text{Inflection Points } \left(\pm 1, \frac{1}{\sqrt{2\pi e}}\right)}$$