Other Exponential and Logarithmic Functions - Differentiation and Integration

Differentiation

$\frac{d}{dx}\left(a^x\right)=a^x\ln a$

$\frac{d}{dx}\left(a^u\right)=a^u\ln a\bullet\frac{du}{dx}$

$\frac{d}{dx}\left(\log_a x\right)=\frac{1}{x\ln a}$

$\frac{d}{dx}\left(\log_a u\right)=\frac{1}{u\ln a}\bullet\frac{du}{dx}$

Integration

$\int a^x dx=\frac{a^x}{\ln a}+C$

$a^x=e^{x\ln a}$

$\frac{d}{dx}(a^x)=\frac{d}{dx}(e^{x\ln a})$

$\frac{d}{dx}(a^x)=e^{x\ln a}\cdot\ln a$

$\boxed{\frac{d}{dx}(a^x)=a^x\ln a}$

$\log_a x=\frac{\ln x}{\ln a}$

$\frac{d}{dx}(\log_a x)=\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$

$\frac{d}{dx}(\log_a x)=\frac{1}{\ln a}\cdot\frac{1}{x}$

$\boxed{\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}}$

Directions: For questions 1 through 11, find the derivative.

1. $y=e^x$

$\boxed{y'=e^x}$

2. $y=e^e$

$\boxed{y'=0}$

3. $y = x^e$

$\boxed{y' = e x^{e-1}}$

4. $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \cdot \ln x$

$\frac{1}{y} y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$\cancel{y} \cdot \frac{1}{\cancel{y}} y' = \ln x + 1 \cdot y$

$y' = (\ln x + 1) y$

$\boxed{y' = (\ln x + 1) x^x}$

5. $y = 6^x$

$\boxed{y' = 6^x \ln 6}$

6. $y = x^2 \cdot 2^x$

$y' = 2x \cdot 2^x + x^2 \cdot 2^x \cdot \ln 2$

$\boxed{y' = 2^x (2x + x^2 \ln 2)}$

7. $y = 6^{-\frac{x}{2}} \cdot \sin(3x+1)$

$$y' = 6^{-\frac{x}{2}} \cdot \ln 6 \cdot -\frac{1}{2} \cdot \sin(3x+1) + 6^{-\frac{x}{2}} \cdot \cos(3x+1) \cdot (3)$$

$$y' = 6^{-\frac{x}{2}} \left[ -\frac{1}{2}\ln 6 \cdot \sin(3x+1) + 3\cos(3x+1) \right]$$

LCD=2

$$y' = \frac{1}{6^{x/2}} \left[ \frac{-\ln 6 \cdot \sin(3x+1) + 6\cos(3x+1)}{2} \right]$$

$$\boxed{y' = \frac{6\cos(3x+1) - \ln 6 \cdot \sin(3x+1)}{2 \cdot 6^{x/2}}}$$

8. $y = \log \frac{x\sqrt{1-2x}}{\sqrt{3x}}$

$$y = \log x + \frac{1}{2}\log(1-2x) - \frac{1}{2}\log(3x)$$

$$y' = \frac{1}{x \ln 10} + \frac{1}{2} \cdot \frac{1}{(1-2x)\ln 10} \cdot -2 - \frac{1}{2} \cdot \frac{1}{3x \ln 10} \cdot 3$$

$$y' = \frac{1}{x \ln 10}^{\cdot 2 \cdot (1-2x)} - \frac{1}{(1-2x)\ln 10}^{\cdot 2x} - \frac{1}{2x \ln 10}^{\cdot (1-2x)}$$

LCD $= 2x(1-2x)\ln 10$

$$y' = \frac{2(1-2x) - 2x - (1-2x)}{2x(1-2x)\ln 10}$$

$$y' = \frac{2 - 4x - 2x - 1 + 2x}{2x(1-2x)\ln 10}$$

$$\boxed{y' = \frac{1-4x}{2x(1-2x)\ln 10}}$$

9. $y=(1+x)^{\frac{1}{x}}$

$\ln y = \ln(1+x)^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln(1+x)$

$\frac{1}{y}y' = -\frac{1}{x^2}\cdot\ln(1+x) + \frac{1}{x}\cdot\frac{1}{1+x}$

$\frac{1}{y}y' = \frac{-\ln(1+x)}{x^2}\cdot(1+x) + \frac{1\cdot x}{x(1+x)}$ LCD $= x^2(1+x)$

$\cancel{y}\cdot\frac{1}{\cancel{y}}y' = \frac{-(1+x)\ln(1+x)+x}{x^2(1+x)}\cdot y$

$$\boxed{y' = \frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\cdot(1+x)^{1/x}}$$

10. $y=\ln x^{\cos x}$

$y = \cos x\cdot\ln x$

$y' = -\sin x\cdot\ln x + \cos x\cdot\frac{1}{x}$

$y' = \frac{-x\sin x\ln x + \cos x}{x}$

$$\boxed{y' = \frac{\cos x - x\sin x\ln x}{x}}$$

11. $y = (\ln x)^{\cos x}$

$$\ln y = \ln (\ln x)^{\cos x}$$

$$\ln y = \cos x \cdot \ln(\ln x)$$

$$\frac{1}{y} \cdot y' = -\sin x \cdot \ln(\ln x) + \cos x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$\frac{1}{y} y' = -\sin x \cdot \ln(\ln x) + \frac{\cos x}{x \ln x} \qquad LCD = x \ln x$$

$$\cancel{y} \cdot \frac{1}{\cancel{y}} y' = \frac{-x \cdot \ln x \cdot \sin x \cdot \ln(\ln x) + \cos x}{x \ln x} \cdot y$$

$$\boxed{y' = \frac{\cos x - x \ln x \sin x \cdot \ln(\ln x)}{x \ln x} \cdot (\ln x)^{\cos x}}$$

Directions: For questions 12 through 15, evaluate the definite integral.

12. $\int_{-3}^{2} 6^x - 2^x \, dx$ $= \left.\frac{6^x}{\ln 6} - \frac{2^x}{\ln 2}\right]_{-3}^{2} = \left(\frac{6^2}{\ln 6} - \frac{2^2}{\ln 2}\right) - \left(\frac{6^{-3}}{\ln 6} - \frac{2^{-3}}{\ln 2}\right)$

$$= \frac{36 \cdot 216 \cdot \ln 2}{\ln 6} - \frac{4 \cdot 216 \cdot \ln 6}{\ln 2} - \frac{1 \cdot \ln 2}{216 \cdot \ln 6} + \frac{1 \cdot 27 \cdot \ln 6}{8 \cdot \ln 2} \qquad LCD = 216 \cdot \ln 6 \cdot \ln 2$$

$$= \frac{7776 \ln 2 - 864 \ln 6 - \ln 2 + 27 \ln 6}{216 \cdot \ln 6 \cdot \ln 2}$$

$$= \boxed{\frac{7775 \ln 2 - 837 \ln 6}{216 \cdot \ln 6 \cdot \ln 2}}$$

13. $\displaystyle\int_{\pi/3}^{\pi/2} 2^{\cos x}\sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$$= -\int_{\pi/3}^{\pi/2} 2^u\,du = \frac{-2^u}{\ln 2} = \left.\frac{-2^{\cos x}}{\ln 2}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{-1}{\ln 2}\left[2^{\cos \pi/2} - 2^{\cos \pi/3}\right] = \frac{-1}{\ln 2}\left[2^0 - 2^{\frac{1}{2}}\right]$$

$$= \frac{-1}{\ln 2}\left[1 - \sqrt{2}\right] = \frac{-1+\sqrt{2}}{\ln 2} = \boxed{\frac{\sqrt{2}-1}{\ln 2}}$$

14. $\displaystyle\int_{1}^{10} \frac{\log_{10} x}{x}\,dx$

$u = \log_{10} x$

$du = \frac{1}{x\ln 10}\,dx$

$\ln 10\,du = \frac{1}{x}\,dx$

$$= \ln 10\int_1^{10} u\,du = \ln 10 \cdot \frac{u^2}{2} = \left.\frac{\ln 10}{2}\left(\log_{10} x\right)^2\right]_1^{10}$$

$$= \frac{\ln 10}{2}\left[(\log_{10} 10)^2 - (\log_{10} 1)^2\right]$$

$$= \frac{\ln 10}{2}\left[1 - 0\right] = \boxed{\frac{\ln 10}{2}}$$

15. $\displaystyle\int_{e}^{10} \frac{1}{x\log_{10} x}\,dx$

$u = \log_{10} x$

$du = \frac{1}{x\ln 10}\,dx$

$\ln 10\,du = \frac{1}{x}\,dx$

$$= \ln 10\int_e^{10} \frac{1}{u}\,du = \ln 10 \cdot \ln|u| = \left.\ln 10 \cdot \ln(\log_{10} x)\right)_e^{10}$$

$$= \ln 10\left[\ln(\log_{10} 10) - \ln(\log_{10} e)\right]$$

$$= \ln 10\left[\underbrace{\ln 1}_{0} - \ln\left(\frac{\ln e}{\ln 10}\right)\right] = \ln 10\left[-\ln(\underbrace{\ln e}_{1}) + \ln(\ln 10)\right]$$

$$= \ln 10\left[\underbrace{-\ln 1}_{0} + \ln(\ln 10)\right] = \boxed{\ln 10 \cdot \ln(\ln 10)}$$

Directions: For questions 16 and 17, find the limit.

16. $\lim_{x\to 0^+} x^x$

$$y = \lim_{x\to 0^+} x^x$$

$$\ln y = \lim_{x\to 0^+} \ln x^x$$

$$\ln y = \lim_{x\to 0^+} x \ln x$$

$$\ln y = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

L'Hopital's Rule: $\ln y = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$$\ln y = \lim_{x\to 0^+} -x$$

$$\ln_e y = 0$$

$$e^0 = y$$

$$\boxed{y = 1}$$

17. $\lim_{x\to\infty} \ln x^{\frac{1}{x}}$

$$y = \lim_{x\to\infty} \ln x^{\frac{1}{x}}$$

$$y = \lim_{x\to\infty} \frac{1}{x} \ln x$$

$$y = \lim_{x\to\infty} \frac{\ln x}{x}$$

L'Hopital's Rule: $y = \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$

$$y = \lim_{x\to\infty} \frac{1}{x}$$

$$\boxed{y = 0}$$