Inverse Trigonometric Functions - Integration

Integration

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a}+C$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a}+C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}+C$$

Directions: Evaluate the integral.

1. $\int \frac{8}{16+x^2}\,dx = 8\int \frac{1}{a^2+u^2}\,du = 8\cdot\frac{1}{4}\cdot\arctan\frac{x}{4} = \boxed{2\arctan\frac{x}{4}+C}$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a}+C$$

$a=4$

$u=x$

$du=dx$

2. $\int \frac{1}{x\sqrt{9x^2-1}}\,dx = \frac{1}{3}\int \frac{1}{\frac{u}{3}\sqrt{u^2-a^2}}\,du = \int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{1}\operatorname{arcsec}\frac{|3x|}{1}$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}+C$$

$= \boxed{\operatorname{arcsec}|3x|+C}$

$a=1$

$u=3x$

$x=\frac{u}{3}$

$du=3dx$

$\frac{1}{3}du=dx$

3. $\int \frac{e^{2x}}{4+e^{4x}}dx = \frac{1}{2}\int \frac{1}{a^2+u^2}\,du = \frac{1}{2}\cdot\frac{1}{2}\cdot \arctan \frac{e^{2x}}{2}$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$

$$\boxed{= \frac{1}{4}\arctan\frac{e^{2x}}{2} + C}$$

$a=2$
$u=e^{2x}$
$du = 2e^{2x}dx$
$\frac{1}{2}du = e^{2x}dx$

4. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}dx = 2\int \frac{1}{\sqrt{a^2-u^2}}\,du = 2\arcsin\frac{\sqrt{x}}{1} = \boxed{2\arcsin\sqrt{x} + C}$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$

$a=1$
$u=\sqrt{x}$
$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

5. $\int \frac{x}{\sqrt{49-(x-7)^2}}dx = \int \frac{u+7}{\sqrt{a^2-u^2}}\,du = \int \frac{u}{\sqrt{a^2-u^2}}\,du + \int \frac{7}{\sqrt{a^2-u^2}}\,du$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$

$v = a^2-u^2$
$dv = -2u\,du$
$-\frac{1}{2}dv = u\,du$

$a=7$
$u=x-7$
$x=u+7$
$du=dx$

$-\frac{1}{2}\int \frac{1}{\sqrt{v}}\,dv + 7\int \frac{1}{\sqrt{a^2-u^2}}\,du$

$-\frac{1}{2}\cdot 2\sqrt{v} + 7\arcsin\frac{x-7}{7}$

$-\sqrt{a^2-u^2} + 7\arcsin\frac{x-7}{7}$

$49-(x-7)^2$
$49-(x^2-14x+49)$
$49-x^2+14x-49$
$-x^2+14x$

$$\boxed{\begin{array}{l} -\sqrt{49-(x-7)^2} + 7\arcsin\frac{x-7}{7} + C \\ \text{OR } -\sqrt{-x^2+14x} + 7\arcsin\frac{x-7}{7} + C \end{array}}$$

6. $\displaystyle\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \int_0^{1/2} u\,du = \frac{u^2}{2} = \left.\frac{(\arcsin x)^2}{2}\right]_0^{1/2}$

$u = \arcsin x$

$du = \dfrac{1}{\sqrt{1-x^2}}\,dx$

$$= \frac{1}{2}\left[(\arcsin \tfrac{1}{2})^2 - (\arcsin 0)^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{6}\right)^2 - (0)^2\right] = \frac{1}{2}\cdot\frac{\pi^2}{36} = \boxed{\frac{\pi^2}{72}}$$

7. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_0^{\pi/2} \frac{1}{a^2+u^2}\,du = -\frac{1}{1}\arctan\frac{\cos x}{1}$

$\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$

$a = 1$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$$= -\arctan(\cos x)\Big]_0^{\pi/2}$$

$$= -\left[\arctan(\underbrace{\cos\tfrac{\pi}{2}}_{0}) - \arctan(\underbrace{\cos 0}_{1})\right]$$

$$= -\left[\arctan 0 - \arctan 1\right]$$

$$= -\left[0 - \pi/4\right] = \boxed{\frac{\pi}{4}}$$

8. $\int \frac{1}{x^2-4x+8}dx$ $= \int \frac{1}{(x-2)^2+4}\,dx = \int \frac{1}{u^2+a^2}\,du = \boxed{\frac{1}{2}\arctan\frac{x-2}{2}+C}$

$(x^2-4x+\underline{4})+8-\underline{4}$

$(x-2)^2+4$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a}+C$

$a=2$

$u=x-2$

$du=dx$

9. $\int \frac{2x}{x^2+10x+41}dx$ $= 2\int \frac{x}{(x+5)^2+16}\,dx = 2\int \frac{u-5}{u^2+a^2}\,du$

$(x^2+10x+\underline{25})+41-\underline{25}$

$(x+5)^2+16$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a}+C$

$u=x+5$

$x=u-5$

$du=dx$

$a=4$

$2\int \frac{u}{u^2+a^2}\,du - 10\int \frac{1}{u^2+a^2}\,du$

$v=u^2+a^2$

$dv=2u\,du$

$\frac{1}{2}dv=u\,du$

$\int \frac{1}{v}\,dv - 10\int \frac{1}{u^2+a^2}\,du$

$\ln|v| - 10\cdot\frac{1}{4}\arctan\frac{x+5}{4}$

$\ln(u^2+a^2) - \frac{5}{2}\arctan\frac{x+5}{4}$

$$\boxed{\begin{gathered}\ln\left[(x+5)^2+16\right]-\frac{5}{2}\arctan\frac{x+5}{4}\\ \text{OR}\\ \ln(x^2+10x+41)-\frac{5}{2}\arctan\frac{x+5}{4}+C\end{gathered}}$$

$(x+5)^2+16$

$x^2+10x+25+16$

$x^2+10x+41$

10. $\int \frac{3x-1}{\sqrt{4x-x^2}}dx = \int \frac{3x-1}{\sqrt{4-(x-2)^2}}dx = \int \frac{3(u+2)-1}{\sqrt{a^2-u^2}}du$

$4x-x^2$
$-x^2+4x$
$-(x^2-4x+\underline{4}) - \underline{-4}$
$-(x-2)^2+4$
$4-(x-2)^2$

$\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\frac{u}{a}+C$

$a=2$
$u=x-2$
$x=u+2$
$du=dx$

$\int \frac{3u+5}{\sqrt{a^2-u^2}}du = 3\int \frac{u}{\sqrt{a^2-u^2}}du + 5\int \frac{1}{\sqrt{a^2-u^2}}du$

$v=a^2-u^2$
$dv=-2u\,du$
$-\frac{1}{2}dv=u\,du$

$-\frac{3}{2}\int \frac{1}{\sqrt{v}}dv + 5\int \frac{1}{\sqrt{a^2-u^2}}du$

$-\frac{3}{2}\cdot 2\sqrt{v} + 5\cdot\arcsin\frac{x-2}{2}$

$-3\sqrt{a^2-u^2} + 5\arcsin\frac{x-2}{2}$

$4-(x-2)^2$
$4-(x^2-4x+4)$
$4-x^2+4x-4$
$-x^2+4x$

$$\boxed{\begin{array}{c} -3\sqrt{4-(x-2)^2}+5\arcsin\frac{x-2}{2}+C \\ \text{OR} \\ -3\sqrt{4x-x^2}+5\arcsin\frac{x-2}{2}+C \end{array}}$$

11. $\int \frac{x}{\left(x^2+1\right)^2+1}\,dx = \frac{1}{2}\int \frac{1}{u^2+a^2}\,du = \frac{1}{2}\cdot\frac{1}{1}\cdot \arctan\frac{x^2+1}{1}$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a}+C$

$\boxed{= \frac{1}{2}\arctan(x^2+1)+C}$

$a=1$

$u=x^2+1$

$du=2x\,dx$

$\frac{1}{2}du = x\,dx$

12. $\int \frac{x+9}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx + 9\int \frac{1}{x^2+9}\,dx$

$u=x^2+9$

$du=2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a}+C$

$a=3$

$u=x$

$du=dx$

$\frac{1}{2}\int \frac{1}{u}\,du + 9\int \frac{1}{a^2+u^2}\,du$

$\frac{1}{2}\ln|u| + 9\cdot\frac{1}{3}\arctan\frac{x}{3}$

$\boxed{\frac{1}{2}\ln(x^2+9) + 3\arctan\frac{x}{3}+C}$

13. $\int \frac{x}{\sqrt{1-x^4}}dx$ $= \frac{1}{2}\int \frac{1}{\sqrt{a^2-u^2}}du = \frac{1}{2}\arcsin\frac{x^2}{1} = \boxed{\frac{1}{2}\arcsin x^2 + C}$

$\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\frac{u}{a} + C$

$a=1$
$u=x^2$
$du=2x\,dx$
$\frac{1}{2}du = x\,dx$

14. $\int \frac{x}{x^4+6x^2+10}dx$ $= \int \frac{x}{(x^2+3)^2+1}dx$

$(x^4+6x^2+\underline{9})+10-\underline{9}$

$(x^2+3)^2+1$

$\int \frac{1}{a^2+u^2}du = \frac{1}{a}\arctan\frac{u}{a} + C$

$a=1$
$u=x^2+3$
$du=2x\,dx$
$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int \frac{1}{u^2+a^2}du = \frac{1}{2}\cdot\frac{1}{1}\cdot\arctan\frac{x^2+3}{1} = \boxed{\frac{1}{2}\arctan(x^2+3) + C}$