Hyperbolic Functions

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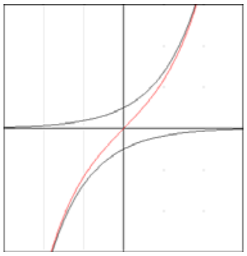
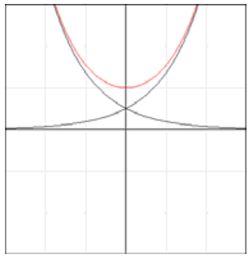


$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

$$\frac{e^x}{2} - \frac{e^{-x}}{2}$$

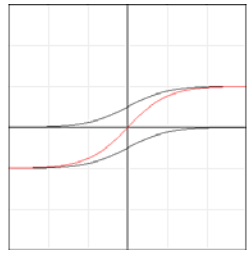


$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Domain: $(-\infty, \infty)$

Range: $[1, \infty)$

$$\frac{e^x}{2} + \frac{e^{-x}}{2}$$

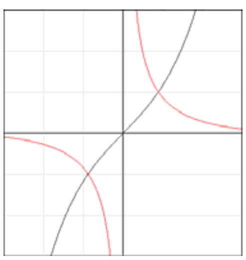


$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Domain: $(-\infty, \infty)$

Range: $(-1, 1)$

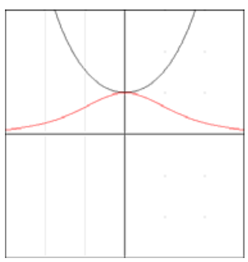
$$\frac{e^x}{e^x + e^{-x}} - \frac{e^{-x}}{e^x + e^{-x}}$$



$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Domain: $(-\infty, 0) \cup (0, \infty)$

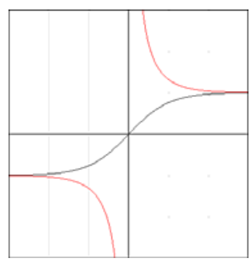
Range: $(-\infty, 0) \cup (0, \infty)$



$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Domain: $(-\infty, \infty)$

Range: $(0, 1]$



$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, -1) \cup (1, \infty)$

Hyperbolic Identities

$\cosh^2 x - \sinh^2 x = 1$

$\tanh^2 x + \operatorname{sech}^2 x = 1$

$\coth^2 x - \operatorname{csch}^2 x = 1$

$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$\sinh 2x = 2\sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x-y) = \cosh x \cosh y + \sinh x \sinh y$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Integrals of Hyperbolic Functions

$$\int \cosh x \; dx = \sinh x + C$$

$$\int \sinh x \; dx = \cosh x + C$$

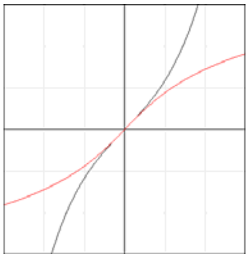
$$\int \operatorname{sech}^2 x \; dx = \tanh x + C$$

$$\int \operatorname{csch} x \coth x \; dx = -\operatorname{csch} x + C$$

$$\int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch}^2 x \; dx = -\coth x + C$$

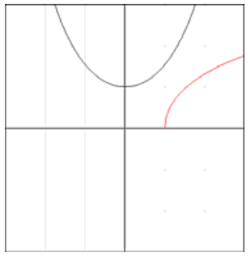
Inverse Hyperbolic Functions



$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

Domain: $(-\infty, \infty)$

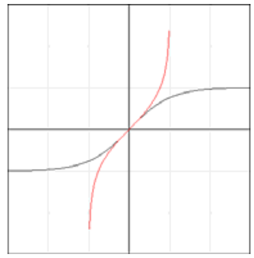
Range: $(-\infty, \infty)$



$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

Domain: $[1, \infty)$

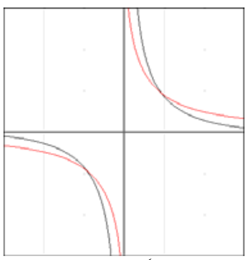
Range: $[0, \infty)$



$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Domain: $(-1, 1)$

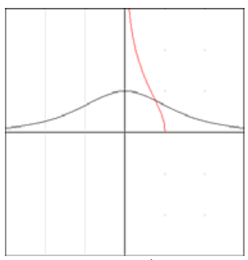
Range: $(-\infty, \infty)$



$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$

Domain: $(-\infty, 0) \cup (0, \infty)$

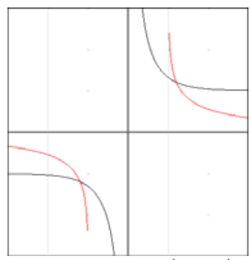
Range: $(-\infty, 0) \cup (0, \infty)$



$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

Domain: $(0, 1]$

Range: $[0, \infty)$



$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

Domain: $(-\infty, -1) \cup (1, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$$

Integrals of Inverse Hyperbolic Functions

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + C$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$$

$$\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C$$

$$\int \frac{1}{x\sqrt{1+x^2}}\,dx = -\operatorname{csch}^{-1}|x| + C$$

$$\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1}|x| + C$$

$$\int \frac{1}{1-x^2}\,dx = \coth^{-1} x + C$$

Directions: For questions 1 through 6, find the value of each expression.

1. $\sinh 0$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\sinh 0 = \frac{e^{0} - e^{-0}}{2}$$

$$\sinh 0 = \frac{1-1}{2}$$

$$\sinh 0 = \boxed{0}$$

2. $\cosh 0$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\cosh x = \frac{e^{0} + e^{-0}}{2}$$

$$\cosh x = \frac{1+1}{2}$$

$$\cosh x = \boxed{1}$$

3. tanh 0

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\tanh x = \frac{0}{1}$$

$$\tanh x = \boxed{0}$$

4. $\sinh^{-1} 0$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$0 = \frac{e^{x} - e^{-x}}{2}$$

$$e^{x} - e^{-x} = 0$$

$$e^{x} - \frac{1}{e^{x}} = 0$$

$$\frac{e^{2x} - 1}{e^{x}} = 0$$

$$e^{2x} - 1 = 0$$

$$e^{2x} = 1$$

$$\ln e^{2x} = \ln 1$$

$$2x = 0$$

$$\boxed{x = 0}$$

OR

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\sinh^{-1} 0 = \ln\left(0 + \sqrt{0^2+1}\right)$$

$$\sinh^{-1} 0 = \ln 1$$

$$\sinh^{-1} 0 = \boxed{0}$$

5. $\cosh^{-1} 0$

$\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

$0 = \dfrac{e^{x}+e^{-x}}{2}$

$0 = e^{x}+e^{-x}$

$0 = \dfrac{e^{2x}+1}{e^{x}}$

$e^{2x}+1=0$

$e^{2x}=-1$

$\ln e^{2x} = \ln -1$

$\boxed{\text{Not possible}}$

OR

$\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$

$\cosh^{-1} 0 = \ln\left(0+\sqrt{0^2-1}\right)$

$\cosh^{-1} 0 = \ln\sqrt{-1}$

$\boxed{\text{Not possible}}$

6. $\tanh^{-1} 0$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

$\tanh x = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$0 = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$e^{x}-e^{-x}=0$

$\dfrac{e^{2x}-1}{e^{x}}=0$

$e^{2x}-1=0$

$e^{2x}=1$

$\ln e^{2x}=\ln 1$

$2x=0$

$\boxed{x=0}$

OR

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\tanh^{-1} 0 = \frac{1}{2}\ln\left(\frac{1+0}{1-0}\right)$

$\tanh^{-1} 0 = \frac{1}{2}\ln 1$

$\tanh^{-1} 0 = \boxed{0}$

Directions: For questions 7 and 8, verify each identity.

7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh^2 x = \frac{\frac{e^{2x}+e^{-2x}}{2} - 1}{2}$$

$$\sinh^2 x = \frac{\frac{e^{2x}+e^{-2x}-2}{2}}{2}$$

$$\sinh^2 x = \frac{e^{2x}-2+e^{-2x}}{4}$$

$$\sinh^2 x = \frac{(e^x - e^{-x})^2}{4}$$

$$\sinh^2 x = \frac{e^x - e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2}$$

$$\sinh^2 x = \sinh^2 x \checkmark$$

8. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$$\sinh(x+y) = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$\sinh(x+y) = \frac{e^{x+y} + e^{x-y} - e^{y-x} - e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}}{4}$$

$$\sinh(x+y) = \frac{2e^{x+y} - 2e^{-(x+y)}}{4}$$

$$\sinh(x+y) = \frac{e^{x+y} - e^{-(x+y)}}{2}$$

$$\sinh(x+y) = \sinh(x+y) \checkmark$$

Directions: For questions 9 through 11, find each limit.

9. $\lim\limits_{x \to \infty} \sinh x$

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{2}$$

$$\lim_{x\to\infty} \frac{e^x - \frac{1}{e^x}}{2}$$

$$\lim_{x\to\infty} \frac{e^{2x}-1}{2e^x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{2e^{2x}}{2e^x} = \lim_{x\to\infty} e^x = e^\infty = \boxed{\infty}$$

10. $\lim\limits_{x \to 0^-} \coth x$

$$\lim_{x\to 0^-} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x\to 0^-} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} = \lim_{x\to 0^-} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}$$

$$\lim_{x\to 0^-} \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^0+1}{e^0-1} = \frac{2}{0}$$

$$\lim_{x\to 0^-} \frac{e^{2x}+1}{e^{2x}-1} = \frac{+}{-} = \boxed{-\infty}$$

11. $\lim\limits_{x \to 0^+} \coth x$

$$\lim_{x\to 0^-} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to 0^-} \frac{e^x+\frac{1}{e^x}}{e^x-\frac{1}{e^x}} = \lim_{x\to 0^-} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}$$

$$\lim_{x\to 0^-} \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^0+1}{e^0-1} = \frac{2}{0}$$

$$\lim_{x\to 0^-} \frac{e^{2x}+1}{e^{2x}-1} = \frac{+}{+} = \boxed{+\infty}$$

Directions: For questions 12 and 13, find the values of the five remaining trigonometric functions.

12. $\sinh x = \frac{4}{3}$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh^2 x - \left(\frac{4}{3}\right)^2 = 1$$

$$\cosh^2 x - \frac{16}{9} = 1$$

$$\cosh^2 x = \frac{25}{9}$$

$$\cosh x = \frac{5}{3}$$

$$\boxed{\begin{array}{ll} \sinh x = \frac{4}{3} & \operatorname{csch} x = \frac{3}{4} \\ \cosh x = \frac{5}{3} & \operatorname{sech} x = \frac{3}{5} \\ \tanh x = \frac{4}{5} & \coth x = \frac{5}{4} \end{array}}$$

13. $\coth x = 2$

$\coth^2 x - \operatorname{csch}^2 x = 1$

$(2)^2 - \operatorname{csch}^2 x = 1$

$4 - \operatorname{csch}^2 x = 1$

$-\operatorname{csch}^2 x = -3$

$\operatorname{csch}^2 x = 3$

$\operatorname{csch} x = \sqrt{3}$

$\sinh x = \frac{1}{\sqrt{3}}$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - \left(\frac{1}{\sqrt{3}}\right)^2 = 1$

$\cosh^2 x - \frac{1}{3} = 1$

$\cosh^2 x = \frac{4}{3}$

$\cosh x = \frac{2}{\sqrt{3}}$

$$\boxed{\begin{array}{ll} \sinh x = \frac{\sqrt{3}}{3} & \operatorname{csch} x = \sqrt{3} \\ \cosh x = \frac{2\sqrt{3}}{3} & \operatorname{sech} x = \frac{\sqrt{3}}{2} \\ \tanh x = \frac{1}{2} & \coth x = 2 \end{array}}$$

Directions: For questions 14 through 18, find each derivative.

14. $y = \sinh 5x$

$y' = \cosh 5x \cdot 5$

$\boxed{y' = 5\cosh 5x}$

15. $y = \ln(\tanh 2x)$

$y' = \frac{1}{\tanh 2x} \cdot \operatorname{sech}^2 2x \cdot 2$

$y' = \coth 2x \cdot \operatorname{sech}^2 2x \cdot 2$

$y' = \frac{\cosh 2x}{\sinh 2x} \cdot \frac{1}{\cosh^2 2x} \cdot 2$

$y' = \frac{2}{\sinh 2x \cosh 2x} = \frac{2}{\frac{\sinh 4x}{2}} = \frac{4}{\sinh 4x} = \boxed{4\operatorname{csch} 4x}$

16. $y = \tan^{-1}(\sinh x)$

$$y' = \frac{1}{1+\sinh^2 x} \cdot \cosh x$$

$$y' = \frac{\cosh x}{\cosh^2 x}$$

$$y' = \frac{1}{\cosh x}$$

$$\boxed{y' = \operatorname{sech} x}$$

17. $y = \tanh^{-1}(\sinh x)$

$$y' = \frac{1}{1-\sinh^2 x} \cdot \cosh x$$

$$\boxed{y' = \frac{\cosh x}{1-\sinh^2 x}}$$

18. $y = (\operatorname{sech}^{-1} 2x)^2$

$$y' = 2(\operatorname{sech}^{-1} 2x) \cdot \frac{-1}{2x\sqrt{1-(2x)^2}} \cdot 2$$

$$y' = \frac{-4(\operatorname{sech}^{-1} 2x)}{2x\sqrt{1-4x^2}}$$

$$\boxed{y' = \frac{-2\operatorname{sech}^{-1} 2x}{x\sqrt{1-4x^2}}}$$

Directions: For questions 19 through 23, evaluate each integral.

19. $\int_0^{\frac{1}{2}} e^x \cosh x \, dx$

$$\int_0^{\frac{1}{2}} e^x \cdot \frac{e^x + e^{-x}}{2} dx = \frac{1}{2}\int_0^{\frac{1}{2}} e^{2x} + 1 \, dx = \frac{1}{2}\int_0^{\frac{1}{2}} e^{2x} dx + \frac{1}{2}\int_0^{\frac{1}{2}} 1 \, dx$$

$u = 2x$
$du = 2dx$
$\frac{1}{2}du = dx$

$$\frac{1}{4}\int_0^{\frac{1}{2}} e^u du + \frac{1}{2}\int_0^{\frac{1}{2}} 1 \, dx = \frac{1}{4}e^u + \frac{1}{2}x = \frac{1}{4}e^{2x} + \frac{1}{2}x \Big]_0^{1/2}$$

$$\left[\frac{1}{4}e^1 + \frac{1}{4}\right] - \left[\frac{1}{4}e^0 + 0\right] = \frac{1}{4}e + \frac{1}{4} - \frac{1}{4} = \boxed{\frac{e}{4}}$$

20. $\int_0^{\ln 3} \tanh x \, dx$

$$\int_0^{\ln 3} \frac{\sinh x}{\cosh x} dx = \int_0^{\ln 3} \frac{1}{u} du = \ln|u| = \ln|\cosh x| \Big]_0^{\ln 3}$$

$u = \cosh x$
$du = \sinh x \, dx$

$$\ln(\cosh \ln 3) - \ln(\cosh 0) = \ln(\cosh \ln 3) - \ln 1$$

$$= \ln[\cosh(\ln 3)] = \ln\left[\frac{e^{\ln 3} + e^{-\ln 3}}{2}\right] = \ln\left[\frac{3 + \frac{1}{3}}{2}\right]$$

$$= \ln\left[\frac{\frac{10}{3}}{2}\right] = \boxed{\ln \frac{5}{3}}$$

21. $\int \cosh^2 x \, dx$

$$\int \frac{\cosh 2x + 1}{2} dx = \frac{1}{2}\int \cosh 2x + 1 \, dx = \frac{1}{2}\int \cosh 2x \, dx + \frac{1}{2}\int 1 \, dx$$

$u = 2x$

$du = 2dx$

$\frac{1}{2}du = dx$

$$\frac{1}{4}\int \cosh u \, du + \frac{1}{2}\int 1 \, dx = \frac{1}{4}\sinh u + \frac{1}{2}x = \boxed{\frac{1}{4}\sinh 2x + \frac{x}{2} + C}$$

22. $\int \frac{1}{16 - x^2} dx$ $= \int \frac{1}{a^2 - u^2} du = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| = \boxed{\frac{1}{8} \ln \left| \frac{4+x}{4-x} \right| + C}$

$a = 4$

$u = x$

$du = dx$

23. $\int \frac{1}{\sqrt{1 - e^{2x}}} dx$ $= \int \frac{1}{u\sqrt{a^2 - u^2}} du = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 - u^2}}{|u|}$

$a = 1$

$u = e^x$

$du = e^x dx$

$\frac{1}{e^x} du = dx$

$\frac{1}{u} du = dx$

$$\boxed{= -\ln \frac{1 + \sqrt{1 - e^{2x}}}{|e^{2x}|} + C}$$