

Partial Fractions

When to use partial fraction decomposition:

1. The integrand is a rational function.
2. The degree of the numerator is less than the degree of the denominator.
3. The denominator factors.

Directions: Integrate using partial fractions.

$$1. \int \frac{2x-1}{x^2-5x+4} dx \quad \frac{2x-1}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} = \frac{\frac{7}{3}}{x-4} + \frac{-\frac{1}{3}}{x-1} = \frac{7}{3(x-4)} - \frac{1}{3(x-1)}$$

$$2x-1 = A(x-1) + B(x-4)$$

$$x=1: \quad 1 = A(0) + B(-3) \quad B = -\frac{1}{3}$$

$$x=4: \quad 7 = A(3) + B(0) \quad A = \frac{7}{3}$$

$$\int \frac{7}{3(x-4)} - \frac{1}{3(x-1)} dx = \frac{7}{3} \int \frac{1}{x-4} dx - \frac{1}{3} \int \frac{1}{x-1} dx$$

$$u = x-4$$

$$du = dx$$

$$u = x-1$$

$$du = dx$$

$$= \frac{7}{3} \int \frac{1}{u} du - \frac{1}{3} \int \frac{1}{u} du$$

$$= \frac{7}{3} \ln|u| - \frac{1}{3} \ln|u|$$

$$= \frac{7}{3} \ln(x-4) - \frac{1}{3} \ln(x-1)$$

=

$$= \boxed{\ln \sqrt[3]{\frac{(x-4)^7}{(x-1)}} + C}$$

$$2. \int \frac{x^3}{x^2+2x+1} dx = \int x-2 + \frac{3x+2}{x^2+2x+1} dx$$

Long Division:

$$\begin{array}{r} x-2 + \frac{3x+2}{x^2+2x+1} \\ \hline x^2+2x+1 \left| \begin{array}{r} x^3 \\ -x^3-2x^2-2x \\ \hline -2x^2-x \\ +2x^2+4x+2 \\ \hline 3x+2 \end{array} \right. \end{array}$$

Partial Fractions:

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$3x+2 = A(x+1) + B$$

$$x=-1: -1 = A(0) + B \quad B = -1$$

$$x=0: 2 = A(1) + -1 \quad A = 3$$

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{3}{x+1} - \frac{1}{(x+1)^2}$$

$$\int x-2 + \frac{3}{x+1} - \frac{1}{(x+1)^2} dx = \int x-2 dx + \int \frac{3}{x+1} - \frac{1}{(x+1)^2} dx$$

$$\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}$$

$$= \int x-2 dx + \int \frac{3}{u} - \frac{1}{u^2} du$$

$$= \frac{x^2}{2} - 2x + 3\ln|u| + \frac{1}{u}$$

$$\boxed{= \frac{x^2}{2} - 2x + 3\ln(x+1) + \frac{1}{x+1} + C}$$

$$3. \int \frac{2}{x^3 + x} dx \quad \frac{2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{2}{x} + \frac{-2x+0}{x^2+1} = \frac{2}{x} - \frac{2x}{x^2+1}$$

$$2 = A(x^2+1) + (Bx+C)(x)$$

$$x=0: \quad 2 = A(1) + C(0) \quad A=2$$

$$x=1: \quad 2 = 2(2) + (B+C)(1) \quad B+C = -2$$

$$x=2: \quad 2 = 2(5) + (2B+C)(2) \quad 2B+C = -4$$

$$B+C = -2$$

$$-(2B+C = -4)$$

$$B+C = -2$$

$$-2B-C = 4$$

$$-B = 2$$

$$B = -2$$

$$B+C = -2$$

$$-2+C = -2$$

$$C = 0$$

$$\int \frac{2}{x} - \frac{2x}{x^2+1} dx = \int \frac{2}{x} dx - \int \frac{2x}{x^2+1} dx$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$\int \frac{2}{x} dx - \int \frac{1}{u} du = 2 \ln|x| - \ln|u|$$

$$= 2 \ln|x| - \ln(x^2+1)$$

$$= \ln x^2 - \ln(x^2+1)$$

$$= \boxed{\ln \frac{x^2}{x^2+1} + C}$$

$$4. \int \frac{1}{(x+2)(x^2+1)} dx = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} = \frac{\frac{1}{5}}{x+2} + \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1} = \frac{1}{5(x+2)} + \frac{-x+2}{5(x^2+1)}$$

$$I = A(x^2+1) + (Bx+C)(x+2)$$

$$x = -2: I = A(5) + (-2B+C)(0) \quad A = 1/5$$

$$x = 0: I = \frac{1}{5}(1) + C(2) \quad C = 2/5$$

$$x = 1: I = \frac{1}{5}(2) + (B+2/5)(3)$$

$$I = \frac{2}{5} + 3B + \frac{6}{5}$$

$$5 = 2 + 15B + 6$$

$$5 = 8 + 15B \quad B = -1/5$$

$$\int \frac{1}{5(x+2)} + \frac{-x+2}{5(x^2+1)} dx = \frac{1}{5} \int \frac{1}{x+2} dx + \frac{1}{5} \int \frac{-x}{x^2+1} dx + \frac{1}{5} \int \frac{2}{x^2+1} dx$$

$$u = x+2$$

$$du = dx$$

$$u = x^2+1$$

$$du = 2x dx$$

$$a = 1$$

$$\frac{1}{2} du = x dx$$

$$u = x$$

$$du = dx$$

$$\frac{1}{5} \int \frac{1}{u} du - \frac{1}{10} \int \frac{1}{u} du + \frac{2}{5} \int \frac{1}{u^2+1} du$$

$$\frac{1}{5} \ln|u| - \frac{1}{10} \ln|u| + \frac{2}{5} \cdot \frac{1}{a} \arctan \frac{u}{a}$$

$$\frac{1}{5} \ln|x+2| - \frac{1}{10} \ln(x^2+1) + \frac{2}{5} \arctan x$$

$$\frac{2 \ln|x+2| - \ln(x^2+1) + 4 \arctan x}{10}$$

$$\boxed{\frac{\ln \frac{(x+2)^2}{(x^2+1)} + 4 \arctan x}{10} + C}$$