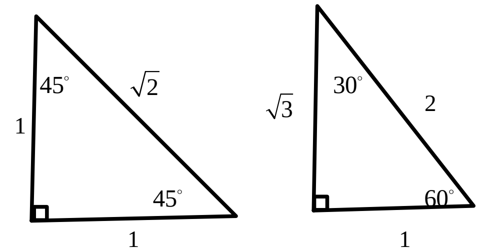
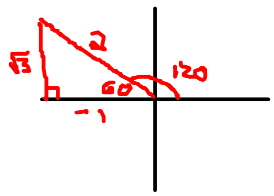
Sum and Difference
Formulas

$$\sin(u+v)=\sin u\cos v+\cos u\sin v$$
$$\sin(u-v)=\sin u\cos v-\cos u\sin v$$
$$\cos(u+v)=\cos u\cos v-\sin u\sin v$$
$$\cos(u-v)=\cos u\cos v+\sin u\sin v$$
$$\tan(u+v)=\frac{\tan u+\tan v}{1-\tan u\tan v}$$
$$\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$$

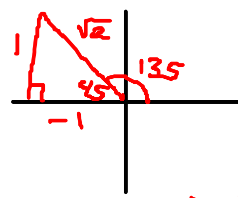
Special Right Triangles



1. Find the exact value of the expression $\sin\left(\underset{u}{\frac{2\pi}{3}}+\underset{v}{\frac{3\pi}{4}}\right)$.



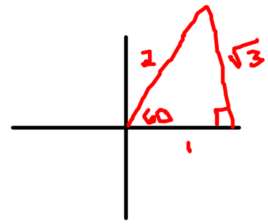
$$\sin(u+v)=\sin u\cos v+\cos u\sin v$$
$$\sin\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)=\sin\frac{2\pi}{3}\cos\frac{3\pi}{4}+\cos\frac{2\pi}{3}\sin\frac{3\pi}{4}$$
$$\frac{2\cancel{\pi}}{3}\cdot\frac{\cancel{180}^{60}}{\cancel{\pi}}=120^\circ$$
$$\frac{3\cancel{\pi}}{\cancel{4}}\cdot\frac{\cancel{180}^{45}}{\cancel{\pi}}=135^\circ$$



$$\sin(120+135)=\sin 120\cos 135+\cos 120\sin 135$$
$$=\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$=\frac{-\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}=\frac{-\sqrt{3}-1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}(-\sqrt{3}-1)}{2\sqrt{4}}=\boxed{\frac{\sqrt{2}(-\sqrt{3}-1)}{4}}$$

2. Use the sum and difference formulas to find the exact value of each.

a) $\sin 105^\circ = \sin(\overset{u}{60} + \overset{v}{45})$

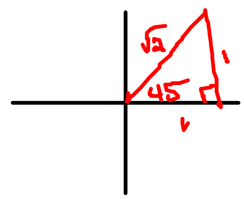


$\sin(u+v) = \sin u \cos v + \cos u \sin v$

$\sin(60+45) = \sin 60 \cos 45 + \cos 60 \sin 45$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$

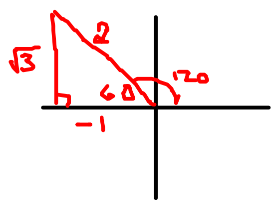
$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$



$= \frac{\sqrt{3}+1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+1)}{2\sqrt{4}}$

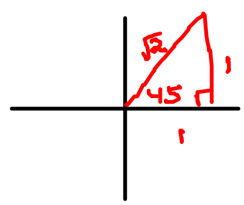
$= \boxed{\frac{\sqrt{2}(\sqrt{3}+1)}{4}}$

b) $\tan 165^\circ = \tan(\overset{u}{120} + \overset{v}{45})$



$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \cdot \tan v}$

$\tan(120+45) = \frac{\tan 120 + \tan 45}{1 - \tan 120 \cdot \tan 45}$



$= \frac{(-\sqrt{3}) + (1)}{1 - (-\sqrt{3})(1)} = \frac{-\sqrt{3}+1}{1+\sqrt{3}}$

$= \frac{1-\sqrt{3}}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1-\sqrt{3}-\sqrt{3}+\sqrt{9}}{1-\cancel{\sqrt{3}}+\cancel{\sqrt{3}}-\sqrt{9}} = \frac{1-2\sqrt{3}+3}{1-3} = \frac{4-2\sqrt{3}}{-2} = \boxed{-2+\sqrt{3}}$

3. Use the sum and difference formulas to write the expression as a function of a single angle.

a) $\underbrace{\sin 340^{\circ} \cos 50^{\circ} - \cos 340^{\circ} \sin 50^{\circ}}$

$\sin(u-v) = \sin u \cos v - \cos u \sin v$

$u = 340 \quad v = 50$

$\sin(340-50) = \boxed{\sin 290}$

b) $\dfrac{\tan 140^{\circ} - \tan 60^{\circ}}{1 + \tan 140^{\circ} \tan 60^{\circ}}$

$\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

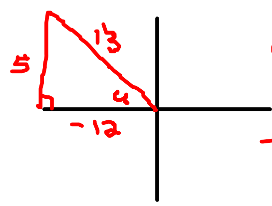
$u = 140 \quad v = 60$

$\tan(140-60) = \boxed{\tan 80}$

4. Find the exact value of $\cos(u-v)$ if:

$\sin u = \dfrac{5}{13}$ where $\dfrac{\pi}{2} < u < \pi$ ($90°$, $180°$)

$\cos v = -\dfrac{3}{5}$ where $\pi < v < \dfrac{3\pi}{2}$ ($180°$, $270°$)

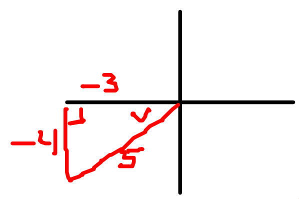


$a^2 + b^2 = c^2$
$5^2 + b^2 = 13^2$
$25 + b^2 = 169$
$-25 \qquad -25$
$b^2 = 144$
$b = 12$

$\cos(u-v) = \cos u \cos v + \sin u \sin v$

$\underbrace{\left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right)} + \underbrace{\left(\frac{5}{13}\right)\left(-\frac{4}{5}\right)}$

$\dfrac{36}{65} + \dfrac{-20}{65} = \boxed{\dfrac{16}{65}}$

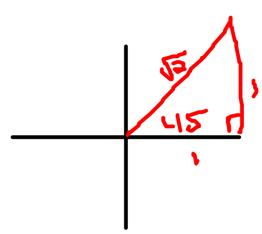


$a^2 + b^2 = c^2$
$(-3)^2 + b^2 = 5^2$
$9 + b^2 = 25$
$-9 \qquad -9$
$b^2 = 16$
$b = 4$

5. Verify the identity.

$\tan\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$ (u, v)

$\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$



$\dfrac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$

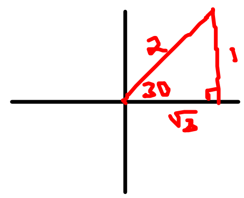
$\dfrac{\pi}{4} \cdot \dfrac{\cancel{180}^{45}}{\pi} = 45°$

$$\frac{1-\tan\theta}{1+(1)\tan\theta}$$

$$\frac{1-\tan\theta}{1+\tan\theta} = \frac{1-\tan\theta}{1+\tan\theta} \checkmark$$

6. Find all solutions of the equation in the interval $0 \le x < 2\pi$.

$$\underbrace{\sin\left(x+\frac{\pi}{6}\right)}-\underbrace{\sin\left(x-\frac{\pi}{6}\right)}=\frac{1}{2}$$

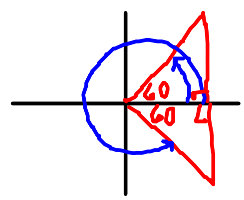


$\sin(u+v) = \sin u\cos v + \cos u \sin v$

$u = x$ $\quad = \sin x\cos 30 + \cos x \sin 30$

$v = \frac{\pi}{6}\cdot\frac{180}{\pi} = 30^\circ$

$= \sin x\left(\frac{\sqrt{3}}{2}\right) + \cos x\left(\frac{1}{2}\right)$



$\sin(u-v) = \sin u\cos v - \cos u \sin v$

$u = x$ $\quad = \sin x\cos 30 - \cos x \sin 30$

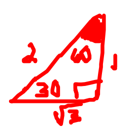
$v = 30^\circ$ $\quad = \sin x\left(\frac{\sqrt{3}}{2}\right) - \cos x\left(\frac{1}{2}\right)$

$\sin x\left(\frac{\sqrt{3}}{2}\right) + \cos x\left(\frac{1}{2}\right) - \left[\sin x\left(\frac{\sqrt{3}}{2}\right) - \cos\left(\frac{1}{2}\right)\right] = \frac{1}{2}$

$\cancel{\sin x\left(\frac{\sqrt{3}}{2}\right)} + \cos x\left(\frac{1}{2}\right) - \cancel{\sin x\left(\frac{\sqrt{3}}{2}\right)} + \cos x\left(\frac{1}{2}\right) = \frac{1}{2}$

$\cos x = \frac{1}{2}$

I $\qquad$ IV



I: $60^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{3}$

IV: $300^\circ \times \frac{\pi}{180} = \frac{5\pi}{3}$

$$\boxed{\{\pi/3, 5\pi/3\}}$$