Double and Half-Angle Formulas

<u>Double-Angle Formulas</u>

$\sin 2u = 2\sin u \cos u$

$\cos 2u = \cos^2 u - \sin^2 u$ or $2\cos^2 u - 1$ or $1 - 2\sin^2 u$

$\tan 2u = \dfrac{2\tan u}{1-\tan^2 u}$

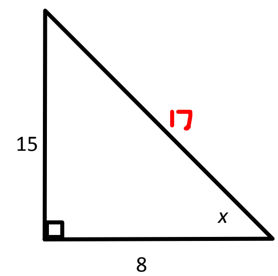
<u>Half-Angle Formulas</u>

$\sin\dfrac{u}{2} = \pm\sqrt{\dfrac{1-\cos u}{2}}$

$\cos\dfrac{u}{2} = \pm\sqrt{\dfrac{1+\cos u}{2}}$

$\tan\dfrac{u}{2} = \dfrac{1-\cos u}{\sin u}$ or $\dfrac{\sin u}{1+\cos u}$

1. Find the exact value of each trigonometric function.



$a^2+b^2=c^2$
$8^2+15^2=c^2$
$64+225=c^2$
$\sqrt{c^2}=\sqrt{289}$
$c=17$

a) $\sin 2x = 2\sin x\cos x = \frac{2}{1}\left(\frac{15}{17}\right)\left(\frac{8}{17}\right) = \boxed{\frac{240}{289}}$

b) $\cos 2x = \cos^2 x - \sin^2 x = \left(\frac{8}{17}\right)^2 - \left(\frac{15}{17}\right)^2$
$= \frac{64}{289} - \frac{225}{289} = \boxed{\frac{-161}{289}}$

c) $\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\left(\frac{15}{8}\right)}{1-\left(\frac{15}{8}\right)^2} = \frac{\frac{15}{4}}{1-\frac{225}{64}} = \frac{\frac{15}{4}}{\frac{64-225}{64}} = \frac{15}{4} \div \frac{-161}{64} = \frac{15}{4}\cdot\frac{64}{-161}$ $\boxed{= \frac{240}{-161}}$

d) $\csc 2x$ $\boxed{\frac{289}{240}}$

e) $\sec 2x$ $\boxed{\frac{289}{-161}}$

f) $\cot 2x$ $\boxed{\frac{-161}{240}}$

g) $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \frac{\sqrt{1-\frac{8}{17}}}{\sqrt{2}} = \frac{\sqrt{\frac{17-8}{17}}}{\sqrt{2}} = \frac{\sqrt{\frac{9}{17}}}{\sqrt{2}} = \frac{3}{\sqrt{17}}\cdot\frac{1}{\sqrt{2}} = \boxed{\frac{3}{\sqrt{34}}}$

h) $\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \frac{\sqrt{\frac{17+8}{17}}}{\sqrt{2}} = \frac{\sqrt{\frac{25}{17}}}{\sqrt{2}} = \frac{5}{\sqrt{17}}\cdot\frac{1}{\sqrt{2}} = \boxed{\frac{5}{\sqrt{34}}}$

i) $\tan\frac{x}{2} = \frac{\frac{3}{\sqrt{34}}}{\frac{5}{\sqrt{34}}} = \boxed{\frac{3}{5}}$

j) $\csc\frac{x}{2}$ $\boxed{\frac{\sqrt{34}}{3}}$

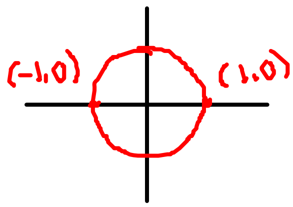
k) $\sec\frac{x}{2}$ $\boxed{\frac{\sqrt{34}}{5}}$

l) $\cot\frac{x}{2}$ $\boxed{\frac{5}{3}}$

2. Find the solutions to each equation for $0 \le x < 2\pi$.

a) $\underbrace{\sin 2x} - \sin x = 0$

$\sin 2x = 2\sin x \cos x$



0, π

$2\sin x\cos x - \sin x = 0$

GCF = $\sin x$

$\sin x(2\cos x - 1) = 0$

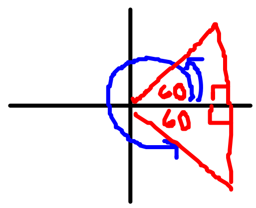
$\sin x = 0$ $\qquad 2\cos x - 1 = 0$

$\qquad\qquad +1 \quad +1$

$\dfrac{2\cos x}{2} = \dfrac{1}{2}$

$\cos x = \frac{1}{2}$

Ref. ∡ = 60°



I: $\cancel{60^\circ} \times \dfrac{\pi}{\cancel{180}} = \dfrac{\pi}{3}$

IV: $\cancel{300^\circ} \times \dfrac{\pi}{\cancel{180}} = \dfrac{5\pi}{3}$

$\boxed{\{0, \pi, \pi/3, 5\pi/3\}}$

b) $\cos 2x + \cos x = 0$

$\cos 2x = 2\cos^2 x - 1$

$2\cos^2 x - 1 + \cos x = 0$

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

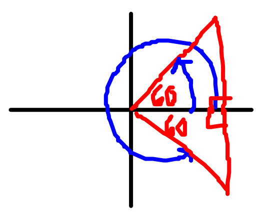
$2\cos x - 1 = 0 \qquad \cos x + 1 = 0$

$\quad +1 \quad +1 \qquad\qquad -1 \quad -1$

$\dfrac{2\cos x}{2} = \dfrac{1}{2} \qquad \cos x = -1$

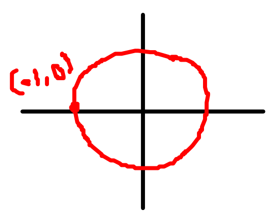
$\cos x = \frac{1}{2}$

Ref. ∡ = 60°



I: $60^\circ \times \dfrac{\pi}{180} = \dfrac{\pi}{3}$

IV: $300^\circ \times \dfrac{\pi}{180} = \dfrac{5\pi}{3}$



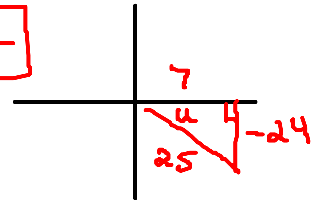
π

$\boxed{\{\pi/3, 5\pi/3, \pi\}}$

3. Find the exact value of each trigonometric function if:

270 360

$$\tan u = -\frac{24}{7} \quad \text{and} \quad \frac{3\pi}{2} < u < 2\pi.$$



a) $\sin 2u = 2\sin u \cos u = \frac{2}{1}\left(-\frac{24}{25}\right)\left(\frac{7}{25}\right) = \boxed{-\frac{336}{625}}$

b) $\cos 2u = \cos^2 u - \sin^2 u = \left(\frac{7}{25}\right)^2 - \left(-\frac{24}{25}\right)^2$

$= \frac{49}{625} - \frac{576}{625} = \boxed{\frac{-527}{625}}$

$a^2+b^2=c^2$

$7^2+(-24)^2=c^2$

$49+576=c^2$

$\sqrt{c^2}=\sqrt{625}$

$c=25$

c) $\tan 2u = \dfrac{\frac{-336}{625}}{\frac{-527}{625}} = \boxed{\frac{336}{527}}$

d) $\sin\frac{u}{2} = -\sqrt{\frac{1-\cos u}{2}} = -\dfrac{\sqrt{\frac{25}{25}\cdot 1 - \frac{7}{25}}}{\sqrt{2}} = -\dfrac{\sqrt{\frac{25-7}{25}}}{\sqrt{2}} = -\frac{\sqrt{18}}{5}\cdot\frac{1}{\sqrt{2}} = \boxed{-\frac{3}{5}}$ $\sqrt{9}$

e) $\cos\frac{u}{2} = +\sqrt{\frac{1+\cos u}{2}} = \dfrac{\sqrt{\frac{25+7}{25}}}{\sqrt{2}} = \frac{\sqrt{32}}{5}\cdot\frac{1}{\sqrt{2}} = \boxed{\frac{4}{5}}$ $\sqrt{16}$

f) $\tan\frac{u}{2} = \dfrac{-\frac{3}{5}}{\frac{4}{5}} = \boxed{-\frac{3}{4}}$

4. Verify each identity.

a) $\csc 2x = \dfrac{\csc x}{2\cos x}$

$\csc 2x = \frac{1}{\sin 2x}$ $\qquad \sin 2x = 2\sin x\cos x$

$\csc x = \frac{1}{\sin x}$

$\frac{1}{2\sin x\cos x} = \frac{1}{2\cos x\cdot\sin x}$

$\frac{1}{2\sin x\cos x} = \frac{1}{2\sin x\cos x}$ ✓

b) $\cos^2 2x - \sin^2 2x = \cos 4x$

$\cos 4x = \cos 4x$ ✓ $\qquad \cos^2 x - \sin^2 x = \cos 2x$

$\cos^2 2x - \sin^2 2x = \cos 4x$

c) $\sec\frac{x}{2} = \pm\sqrt{\frac{2\tan x}{\tan x + \sin x}}$

$\sec\frac{x}{2} = \frac{1}{\cos\frac{x}{2}}$ $\quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

$\tan x = \frac{\sin x}{\cos x}$

$$\frac{1}{\pm\sqrt{\frac{1+\cos x}{2}}} = \pm\sqrt{\frac{2\cdot\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\sin x}{1}\cdot\frac{\cos x}{\cos x}}}$$

$$1\cdot\pm\sqrt{\frac{2}{1+\cos x}} = \pm\sqrt{\frac{\frac{2\sin x}{\cos x}}{\frac{\sin x + \sin x\cos x}{\cancel{\cos x}}}}$$

$$\pm\sqrt{\frac{2}{1+\cos x}} = \pm\sqrt{\frac{2\sin x}{\sin x + \sin x\cos x}} \quad GCF = \sin x$$

$$= \pm\sqrt{\frac{2\cancel{\sin x}}{\cancel{\sin x}(1+\cos x)}}$$

$$\pm\sqrt{\frac{2}{1+\cos x}} = \pm\sqrt{\frac{2}{1+\cos x}} \checkmark$$