Graphs of Rational Functions

$$f(x) = \frac{x^3 + 6x^2 + 11x + 6}{x^2 - 1} = \frac{\cancel{(x+1)}(x+2)(x+3)}{\cancel{(x+1)}(x-1)}$$

VA

$p: \pm 1, \pm 2, \pm 3, \pm 6$

$q: \pm 1$

$p/q: \pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1x^2 & 5x & 6 & \boxed{0} \end{array}$$

$x^2 + 5x + 6$

$(x+3)(x+2)$

Domain

the allowable x-values

$x+1 \neq 0 \qquad x-1 \neq 0$

$$\boxed{x \neq -1 \qquad x \neq 1}$$

Vertical Asymptote

set the denominator equal to zero

$x - 1 = 0$

$$\boxed{\text{VA: } x = 1}$$

Hole/Open Circle/Deleted Point

set the denominator that cancels equal to zero

$x + 1 = 0$

$x = -1$

$$y = \frac{(-1+2)(-1+3)}{(-1-1)}$$

$$y = \frac{(1)(2)}{-2} = -1$$

$$\boxed{\text{Hole: } (-1, -1)}$$

Horizontal Asymptote

Degree of Numerator > Degree of Denominator Horizontal Asymptote: None ✓

Degree of Numerator < Degree of Denominator Horizontal Asymptote: $y = 0$

Degree of Numerator = Degree of Denominator Horizontal Asymptote: $y = \dfrac{\text{Leading Coefficient of Numerator}}{\text{Leading Coefficient of Denominator}}$

$$f(x) = \frac{x^3 + 6x^2 + 11x + 6}{x^2 - 1}$$

$\boxed{\text{NO HA}}$

Deg. Num = 3

Deg. Den = 2

Slant/Oblique Asymptote

use long division only if there is no horizontal asymptote

$$
\begin{array}{r|l}
 & x + 6 \\
\underline{x}^2 - 1 & \underline{\cancel{x^3}} + 6x^2 + 11x + 6 \\
 & -\cancel{x^3} \qquad\qquad \ominus^{+} x \\
\hline
 & \underline{\cancel{6x^2}} + 12x + 6 \\
 & -\cancel{6x^2} \qquad\qquad {}^{+}\ominus 6 \\
\hline
 & 12x + 12
\end{array}
$$

$\frac{x^3}{x^2} = x$

$\frac{6x^2}{x^2} = 6$

$\boxed{\text{SA: } \quad y = x + 6}$

x-intercepts

set y equal to zero and solve for x

$$\frac{(x+2)(x+3)}{(x-1)} = 0$$

$(x+2)(x+3) = 0$

$x + 2 = 0 \qquad x + 3 = 0$

$x = -2 \qquad x = -3$

$\boxed{\begin{array}{l}\text{x.int: } (-2, 0) \\ \qquad\quad (-3, 0)\end{array}}$

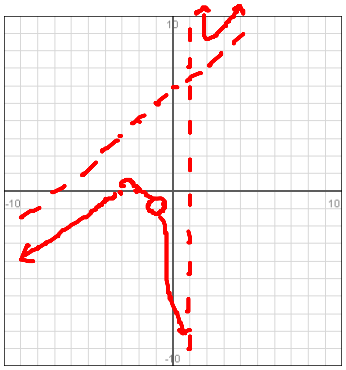
y-intercept

set x equal to zero and solve for y

$$y = \frac{(0+2)(0+3)}{(0-1)}$$

$$y = \frac{2 \cdot 3}{-1} = -6$$

$\boxed{\text{yint: } (0, -6)}$



Steps to Graph Rational Functions

1. Find the Domain.
2. Find all asymptotes.
3. Find all intercepts.

1. Graph each rational function.

a) $f(x)=\frac{x^2+1}{x}$ D: $x \neq 0$

VA: $\boxed{x=0}$ Hole: NONE

HA: Deg Num=2 > Deg Den=1

NONE

SA:

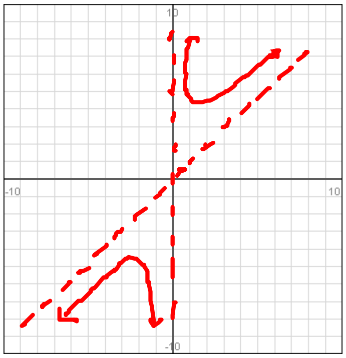
$$x \overline{\big) \cancel{x^2} + 0x + 1}^{\,x}$$

$$\frac{-x^2}{\quad} \downarrow$$

$$0x+1$$

$\boxed{y=x}$

$\frac{x^2}{x} = x$



x-int

$0=\frac{x^2+1}{x}$

$0=x^2+1$

$-1 \quad -1$

$-1=x^2$

$\boxed{\text{NONE}}$

y-int

$y=\frac{0^2+1}{0}$

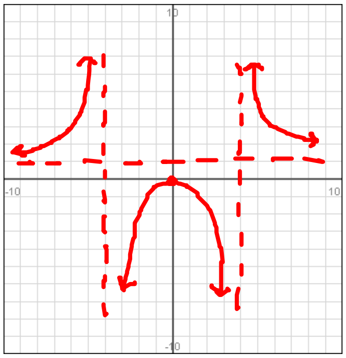
$\boxed{\text{NONE}}$

b) $f(x)=\frac{x^2}{x^2-16}=\frac{x^2}{(x+4)(x-4)}$ D: $x+4 \neq 0$ $x-4 \neq 0$

$\boxed{x \neq -4 \quad x \neq 4}$

VA: $x+4=0$ $x-4=0$

$\boxed{x=-4 \quad x=4}$ Hole: NONE



HA: Deg Num=2 = Deg Den=2

$y=\frac{1}{1}=$ $\boxed{y=1}$

SA: NONE

x-int

$0=\frac{x^2}{x^2-16}$

$0=x^2$

$x=0$ $\boxed{(0,0)}$

y-int

$y=\frac{0^2}{0^2-16}$

$y=0$ $\boxed{(0,0)}$

$f(-1)=\frac{(-1)^2}{(-1)^2-16}$

$=\frac{1}{-15}$

$f(1)=\frac{1^2}{1-16}=\frac{1}{-15}$

c) $f(x)=\dfrac{x}{x^2-1}=\dfrac{x}{(x+1)(x-1)}$

D: $x+1\neq 0$ $\quad x-1\neq 0$

$x\neq -1$ $\quad x\neq 1$

VA: $x+1=0$ $\quad x-1=0$

$\boxed{x=-1 \quad x=1}$

Hole: NONE

HA: Deg Num = 1 < Deg Den = 2

$\boxed{y=0}$

SA: NONE

x-INt

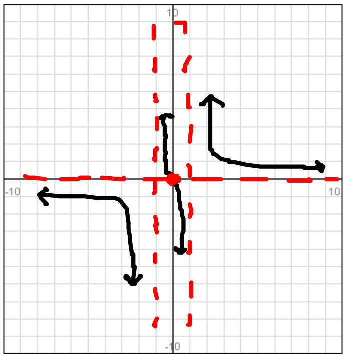
$0=\dfrac{x}{x^2-1}$

$0=x$ $\quad \boxed{(0,0)}$

y INt

$y=\dfrac{0}{0^2-1}$

$y=0$ $\quad \boxed{(0,0)}$



$f(-1/2)=\dfrac{-\frac{1}{2}}{(-\frac{1}{2})^2-1}=\dfrac{-1/2}{-3/4}>0$

$f(1/2)=\dfrac{1/2}{(1/2)^2-1}=\dfrac{1/2}{-3/4}<0$

d) $f(x)=\dfrac{2}{x^2+1}$

D: $x^2+1\neq 0$

$-1 \quad -1$

$x^2=-1$

$\mathbb{R}$

VA: $x^2+1=0$

NONE

Hole: NONE

HA: Deg Num = 0 < Deg Den = 2

$\boxed{y=0}$

SA: NONE

x-INt

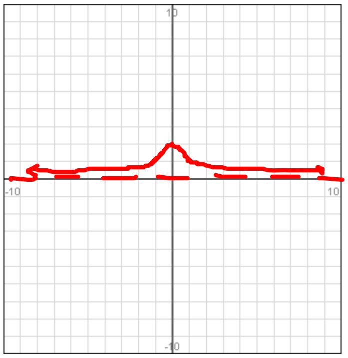
$0=\dfrac{2}{x^2+1}$

$0=2$ $\quad$ NONE

Y-INt

$y=\dfrac{2}{0^2+1}$

$y=2$ $\quad \boxed{(0,2)}$



$f(-1)=\dfrac{2}{(-1)^2+1}=\dfrac{2}{2}=1$

$(-1,1)$

$f(1)=\dfrac{2}{1^2+1}=\dfrac{2}{2}=1$

$(1,1)$

$f(-5)=\dfrac{2}{(-5)^2+1}=\dfrac{2}{26}$