Partial Fraction Decomposition

Partial Fraction Decomposition is used to write a single fraction as a sum or difference of two or more fractions.

$$\frac{5x+1}{x^2+x-2}=\frac{3}{x+2}+\frac{2}{x-1}$$

The degree of the numerator must be less than the degree of the denominator.

Distinct Linear Factors

1. Write the partial fraction decomposition of $\frac{5x+1}{x^2+x-2}$.

$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$\frac{5x+1}{(x+2)(x-1)} = \frac{A\cdot(x-1)}{x+2\cdot(x-1)} + \frac{B\cdot(x+2)}{x-1\cdot(x+2)}$

LCD $=(x+2)(x-1)$

$\frac{5x+1}{(x+2)(x-1)} = \frac{A(x-1)}{(x+2)(x-1)} + \frac{B(x+2)}{(x+2)(x-1)}$

$* \quad 5x+1 = A(x-1)+B(x+2)$

$x-1=0$ $\quad 5(1)+1 = A(1-1)+B(1+2)$

$x=1$ $\quad 6=3B$

$B=2$

$x+2=0$ $\quad 5(-2)+1 = A(-2-1)+B(-2+2)$

$x=-2$ $\quad -9=-3A$

$A=3$

$A=3 \quad B=2$

$\frac{3}{x+2}+\frac{2}{x-1}$

Repeated Linear Factors

2. Write the partial fraction decomposition of $\frac{2x^2+16x+29}{(x+3)^2(x+4)}$.

$= \frac{A}{x+3} + \frac{B}{x+4} + \frac{C}{(x+3)^2}$

$(x+3)(x+3)(x+4)$

$A=5 \quad B=-3 \quad C=-1$

$\frac{2x^2+16x+29}{(x+3)^2(x+4)} = \frac{A\cdot(x+4)}{x+3\cdot(x+4)(x+3)} + \frac{B\cdot(x+3)^2}{x+4\cdot(x+3)^2} + \frac{C\cdot(x+4)}{(x+3)^2\cdot(x+4)}$

$LCD=(x+3)^2(x+4)$

$\frac{5}{x+3} + \frac{-3}{x+4} + \frac{-1}{(x+3)^2}$

$\frac{2x^2+16x+29}{(x+3)^2(x+4)} = \frac{A(x+4)(x+3)}{(x+3)^2(x+4)} + \frac{B(x+3)^2}{(x+3)^2(x+4)} + \frac{C(x+4)}{(x+3)^2(x+4)}$

$* \; 2x^2+16x+29 = A(x+4)(x+3) + B(x+3)^2 + C(x+4)$

$x+3=0$ $\quad 2(-3)^2+16(-3)+29 = A(-3+4)(-3+3) + B(-3+3)^2 + C(-3+4)$

$x=-3$ $\quad -1 = 1C$

$C=-1$

$x+4=0$ $\quad 2(-4)^2+16(-4)+29 = A(-4+4)(-4+3) + B(-4+3)^2 + C(-4+4)$

$x=-4$ $\quad -3 = 1B$

$B=-3$

$x=0$ $\quad 2(0)^2+16(0)+29 = A(0+4)(0+3) + (-3)(0+3)^2 + (-1)(0+4)$

$29 = 12A - 27 - 4$

$29 = 12A - 31$

$+31 \qquad +31$

$\frac{60}{12} = \frac{12A}{12}$

$A=5$

Distinct Quadratic Factors

3. Write the partial fraction decomposition of $\frac{2x-3}{x^3+10x}$.

$\frac{2x-3}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$

$\frac{2x-3}{x(x^2+10)} = \frac{A\cdot(x^2+10)}{x\cdot(x^2+10)} + \frac{Bx+C}{x^2+10}\cdot\frac{x}{x}$

LCD $= x(x^2+10)$

$\frac{2x-3}{x(x^2+10)} = \frac{A(x^2+10)}{x(x^2+10)} + \frac{(Bx+C)(x)}{x(x^2+10)}$

$2x-3 = A(x^2+10) + (Bx+C)(x)$

$2x-3 = Ax^2 + 10A + Bx^2 + Cx$

x^2: $0 = A + B$ $\quad 0 = \frac{-3}{10} + B$ $\quad +\frac{3}{10}\ +\frac{3}{10}$ $\quad \frac{3}{10} = B$

x^1: $2 = C$

x^0: $\frac{-3}{10} = \frac{10A}{10}$ $\quad A = -\frac{3}{10}$

$A = -\frac{3}{10}$ $\quad B = \frac{3}{10}$ $\quad C = 2$

$\frac{\frac{-3}{10}}{x} + \frac{\frac{3}{10}x + \frac{2\cdot 10}{1\cdot 10}}{x^2+10}$

$\frac{\frac{-3}{10}}{\frac{x}{1}} + \frac{\frac{3x}{10} + \frac{20}{10}}{\frac{x^2+10}{1}}$

$-\frac{3}{10}\cdot\frac{1}{x} + \frac{3x+20}{10}\cdot\frac{1}{x^2+10}$

$\boxed{\frac{-3}{10x} + \frac{3x+20}{10(x^2+10)}}$

Repeated Quadratic Factors

4. Write the partial fraction decomposition of $\frac{2x-1}{x(x^2+1)^2}$.

$$\frac{2x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

$x(x^2+1)(x^2+1)$

$A=-1 \quad B=1 \quad C=0 \quad D=1 \quad E=2$

$$\frac{2x-1}{x(x^2+1)^2} = \frac{A\cdot(x^2+1)^2}{x\cdot(x^2+1)^2} + \frac{(Bx+C)\cdot x(x^2+1)}{(x^2+1)\cdot x(x^2+1)} + \frac{(Dx+E)\cdot x}{(x^2+1)^2\cdot x}$$

LCD $= x(x^2+1)^2$

$$\frac{-1}{x} + \frac{1x+0}{(x^2+1)} + \frac{1x+2}{(x^2+1)^2}$$

$$\boxed{\frac{-1}{x} + \frac{x}{(x^2+1)} + \frac{x+2}{(x^2+1)^2}}$$

$$\frac{2x-1}{x(x^2+1)^2} = \frac{A(x^2+1)^2}{x(x^2+1)^2} + \frac{(Bx+C)(x)(x^2+1)}{x(x^2+1)^2} + \frac{(Dx+E)(x)}{x(x^2+1)^2}$$

$$2x-1 = \underbrace{A(x^2+1)(x^2+1)}_{A(x^4+2x^2+1)\;\;Ax^4+2Ax^2+A} + \underbrace{(Bx+C)(x)(x^2+1)}_{(Bx+C)(x^3+x)\;\;Bx^4+Bx^2+Cx^3+Cx} + \underbrace{(Dx+E)(x)}_{Dx^2+Ex}$$

$* \quad 2x-1 = Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$

$x^4: \quad 0 = A+B \qquad 0 = -1+B$ (+1, +1)

$B=1$

$x^3: \quad 0 = C$

$x^2: \quad 0 = 2A+B+D \qquad 0 = 2(-1)+1+D$

$0 = -2+1+D$

$0 = -1+D \longrightarrow D=1$

$x^1: \quad 2 = C+E \qquad 2 = 0+E$

$E=2$

$x^0: \quad -1 = A$

Partial Fraction Decomposition With Long Division

5. Write the partial fraction decomposition of $\frac{x^3-x+3}{x^2+x-2}$.

Degree numerator = 3
Degree Denominator = 2

If degree of numerator is ≥ degree of denominator, use long division

$x^2+x-2 \overline{)\, x^3+0x^2-x+3}$

x

$-x^3 \mp x^2 \pm 2x$

$-x^2+x+3$

$\pm x^2 \mp x \pm 2$

$2x+1$

$\frac{x^3}{x^2} = x$

$\frac{-x^2}{x^2} = -1$

$x-1+\frac{2x+1}{x^2+x-2}$

use partial fraction decomposition

$\frac{2x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

Distinct Linear Factors (same as #1)

$A=1 \quad B=1$

$x-1+\frac{1}{x+2}+\frac{1}{x-1}$

$\frac{2x+1}{(x+2)(x-1)} = \frac{A\cdot(x-1)}{x+2\cdot(x-1)} + \frac{B\cdot(x+2)}{x-1\cdot(x+2)}$

$LCD = (x+2)(x-1)$

$\frac{2x+1}{(x+2)(x-1)} = \frac{A(x-1)}{(x+2)(x-1)} + \frac{B(x+2)}{(x+2)(x-1)}$

$* \quad 2x+1 = A(x-1) + B(x+2)$

$x-1=0$
$x=1$

$2(1)+1 = A(1-1) + B(1+2)$
$3=3B$
$B=1$

$x+2=0$
$x=-2$

$2(-2)+1 = A(-2-1) + B(-2+2)$
$-3=-3A$
$A=1$