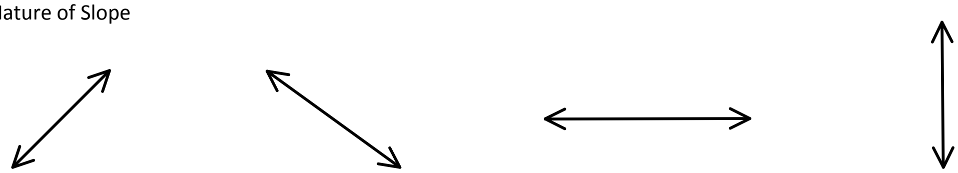
Equations of Tangent and Normal Lines to a Curve

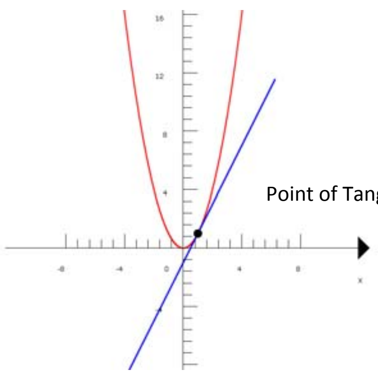
<u>Review:</u>

1. Slope $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

2. Nature of Slope



m > 0 m < 0 m = 0 m = undefined/no slope

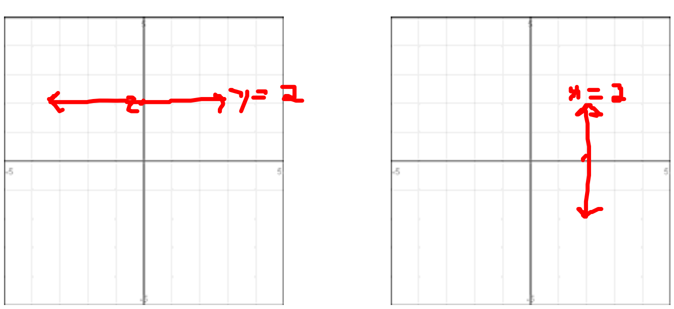


3. Theorems on Slope
   a. If two lines are parallel then their slopes are equal.
   b. If two lines are perpendicular then their slopes are negative reciprocals of each other.

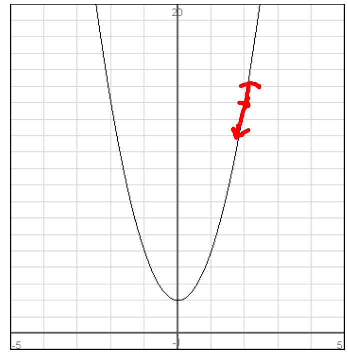
$m = \frac{3}{4}$ $m = -2$ $m = 0$

$m_\perp = -\frac{4}{3}$ $m_\perp = \frac{1}{2}$ $m_\perp =$ undefined

4. Equations of a Line
   a. General Form of a Line: $ax + by + c = 0$ where a, b and c are integers and $a > 0$.
   b. Point-Slope Form of a Line: $y - y_1 = m(x - x_1)$ where m is the slope of the line and (x_1, y_1) is a specific point on the line.
   c. Slope-Intercept Form of a Line: $y = mx + b$ where m is the slope and b is the y-intercept.
   d. Equation of a Horizontal Line: $y = c$ $c = 2$
   e. Equation of a Vertical Line: $x = c$ $c = 2$



1. Find the equation of the line tangent to the curve $y = 3x^2 + 2$ at $(2,14)$ in slope-intercept form.



$y' = 6x \Big|_{(2,14)} = 6(2) = 12$

$x_1 \; y_1$

$m_T = 12 \qquad (2,14)$

$y - y_1 = m(x - x_1)$

$y - 14 = 12(x-2)$

$y - 14 = 12x - 24$

$+14 \qquad +14$

$\boxed{y = 12x - 10}$

2. Find the equation of the line tangent to the curve $xy + y^2 + 2 = 0$ at $y = 1$ in general form.

$(1)y + (x)(1)y' + 2y\,y' + 0 = 0$

$y + xy' + 2y\,y' = 0$

$-y \qquad -y$

$xy' + 2yy' = -y$

$\dfrac{y'(x+2y)}{x+2y} = \dfrac{-y}{x+2y}$

$y' = \dfrac{-y}{x+2y}\Big|_{(-3,1)} = \dfrac{-1}{-3+2(1)} = \dfrac{-1}{-3+2} = \dfrac{-1}{-1} = 1$

$x(1) + 1^2 + 2 = 0$

$x + 1 + 2 = 0$

$x + 3 = 0$

$-3 \quad -3$

$x = -3$

$m_T = 1 \qquad (-3,1)$

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x - -3)$

$y - 1 = x + 3$

$-y + 1 \qquad -y + 1$

$0 = x - y + 4$

$\boxed{x - y + 4 = 0}$

3. Find the equation of the line tangent to the curve $y = \frac{x}{x^2-1}$ at $x = 2$ in general form.

$$y' = \frac{(1)(x^2-1)-(x)(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$$

$$y'\Big|_{x=2} = \frac{-(2)^2-1}{(2^2-1)^2} = \frac{-4-1}{(4-1)^2} = \frac{-5}{9} \qquad m_T = \frac{-5}{9}$$

$$y = \frac{2}{2^2-1} = \frac{2}{3} \qquad (2, 2/3)$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{2}{3} = \frac{-5}{9}(x-2)$$

$$\frac{9\cdot y}{9\cdot 1} - \frac{2\cdot 3}{3\cdot 3} = -\frac{5}{9}x + \frac{10}{9} \qquad LCD = 9$$

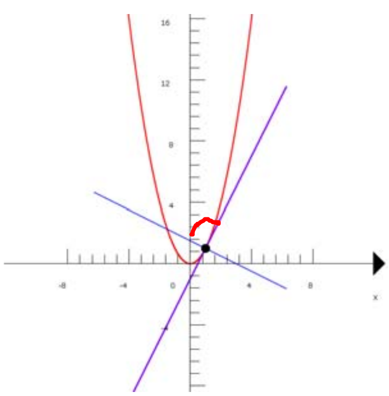
$$\frac{9y}{\cancel{9}} - \frac{6}{\cancel{9}} = \frac{-5x}{\cancel{9}} + \frac{10}{\cancel{9}}$$

$$9y - 6 = -5x + 10$$

$$+5x \quad -10 \qquad +5x \quad -10$$

$$\boxed{5x + 9y - 16 = 0}$$

<u>Normal Line</u> - A normal line is a line that is perpendicular line to the tangent line at the point of tangency.



4. Find the equation of normal line to $y^3 + x^3 - 5y - x^2 + 4 = 0$ at $(1,-3)$ in slope-intercept form.

$3y^2 \cdot y' + 3x^2 - 5y' - 2x + 0 = 0$

$3y^2 y' + 3x^2 - 5y' - 2x = 0$

$\quad -3x^2 \quad +2x \quad -3x^2 + 2x$

$3y^2 y' - 5y' = -3x^2 + 2x$

$\dfrac{y'(3y^2 - 5)}{3y^2 - 5} = \dfrac{-3x^2 + 2x}{3y^2 - 5}$

$y' = \left.\dfrac{-3x^2 + 2x}{3y^2 - 5}\right|_{(1,-3)} = \dfrac{-3(1)^2 + 2(1)}{3(-3)^2 - 5} = \dfrac{-3+2}{27-5} = \dfrac{-1}{22}$

$m_T = \dfrac{-1}{22}$

$m_N = 22$

$m_N = 22 \quad (1,-3)$

$y - y_1 = m(x - x_1)$

$y - -3 = 22(x - 1)$

$y + 3 = 22x - 22$

$\quad -3 \qquad\qquad -3$

$\boxed{y = 22x - 25}$

5. Find the equation of normal line to $f(x)=\sqrt{3x^2-2}$ at $x=3$ in slope-intercept form.

$f(x)=(3x^2-2)^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}(3x^2-2)^{-\frac{1}{2}}(6x)=\frac{\cancel{6x}^{\,3}}{\cancel{2}\sqrt{3x^2-2}}=\frac{3x}{\sqrt{3x^2-2}}$

$f'(3)=\frac{3(3)}{\sqrt{3(3)^2-2}}=\frac{9}{\sqrt{27-2}}=\frac{9}{\sqrt{25}}=\frac{9}{5}$

$m_T=\frac{9}{5}$

$m_N=-\frac{5}{9}$

$f(3)=\sqrt{3(3)^2-2}=\sqrt{27-2}=\sqrt{25}=5$ $(3,5)$

$(3,5)$ $m_N=-\frac{5}{9}$

$y-y_1=m(x-x_1)$

$y-5=-\frac{5}{9}(x-3)$

$y-5=-\frac{5}{9}x+\frac{15}{9}$

$+5 \qquad +5$

$\boxed{y=-\frac{5}{9}x+\frac{20}{3}}$

$\frac{15}{9}+\frac{5\cdot 9}{1\cdot 9}$

$\frac{15}{9}+\frac{45}{9}=\frac{60}{9}$

$\frac{20}{3}$