Relationship Between Continuity and Differentiability

<u>A function is continuous at $x = a$ if:</u>

a) $f(a)$ exists function value

b) $\lim_{x \to a} f(x)$ exists

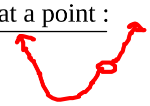
c) $\lim_{x \to a} f(x) = f(a)$

<u>A function is differentiable at $x = a$ if:</u>

a) $f(a)$ is continuous

b) $f'(a^-) = f'(a^+)$

<u>Geometric conditions that prevent a function from being differentiable at a point :</u>



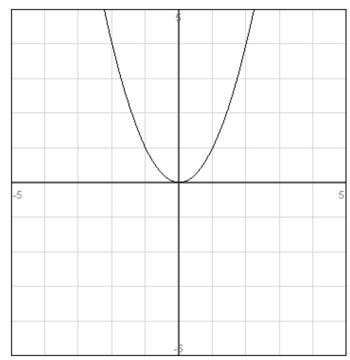
a) Any point of discontinuity (asymptote, deleted point)

b) Corner point 

c) Cusp 

<u>A function that is continuous and differentiable.</u>

$f(x) = x^2$



$\lim_{x \to 0^-} x^2 = 0^2 = 0$

$\lim_{x \to 0^+} x^2 = 0^2 = 0$

$\lim_{x \to 0} x^2 = 0$

$f(0) = 0^2 = 0$

$f(0) = \lim_{x \to 0}$

continuous

$f'(x) = 2x$

$f'(0^-) = 2(0) = 0$

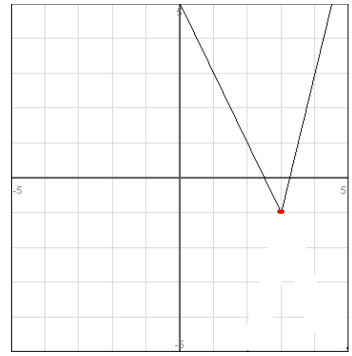
$f'(0^+) = 2(0) = 0$

$f'(0) = 0$

differentiable at $x = 0$

A function that is continuous but not differentiable.

$$f(x)=\begin{cases}5-2x & \text{for } x<3 \\ 4x-13 & \text{for } x\geq 3\end{cases}$$



$\lim_{x\to 3^-} f(x) = 5-2(3) = -1$

$\lim_{x\to 3^+} f(x) = 4(3)-13 = -1$

$\lim_{x\to 3} f(x) = -1$

$f(3) = 4(3)-13 = -1$

$f(3) = \lim_{x\to 3}$

continuous

$$f'(x)=\begin{cases}-2 & x<3 \\ 4 & x>3\end{cases}$$

$f'(3^-) = -2$

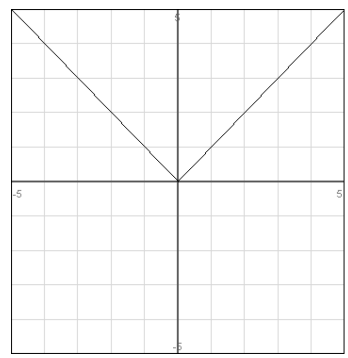
$f'(3^+) = 4$

$f'(3) =$ DNE

not differentiable at $x=3$

A function that is continuous but not differentiable.

$f(x)=|x|$



$\lim_{x\to 0^-} |x| = |0| = 0$

$\lim_{x\to 0^+} |x| = |0| = 0$

$\lim_{x\to 0} |x| = 0$

$f(0) = |0| = 0$

$f(0) = \lim_{x\to 0}$

continuous

$$f'(x)=\begin{cases}1 & x>0 \\ -1 & x<0\end{cases}$$

$f'(0^-) = -1$

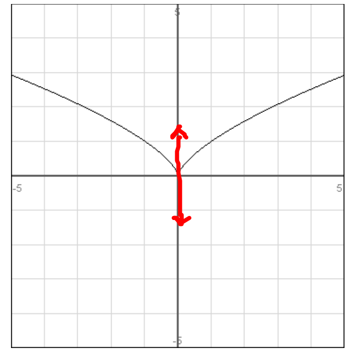
$f'(0^+) = 1$

$f'(0) =$ DNE

not differentiable at $x=0$

A function that is continuous but not differentiable.

$f(x) = x^{\frac{2}{3}}$



$\lim_{x\to 0^-} x^{\frac{2}{3}} = 0^{\frac{2}{3}} = 0$

$\lim_{x\to 0^+} x^{\frac{2}{3}} = 0^{\frac{2}{3}} = 0$

$\lim_{x\to 0} x^{\frac{2}{3}} = 0$

$f(0) = 0^{\frac{2}{3}} = 0$

$f(0) = \lim_{x\to 0}$

continuous

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

$f'(0)_- = \frac{2}{3\sqrt[3]{0}} =$ undef.

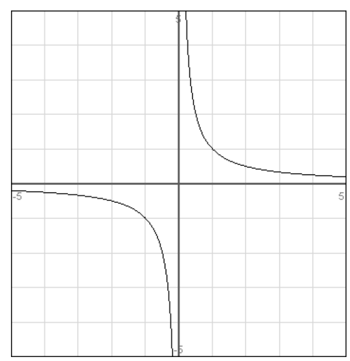
$f'(0)_+ =$ undef.

$f'(0) =$ DNE

not differentiable at $x = 0$

A function that is not continuous and therefore not differentiable.

$f(x) = \frac{1}{x}$



$f(x) = x^{-1}$

$\lim_{x\to 0^-} \frac{1}{x} = \frac{1}{-.1} = -\infty$

$x = -.1$

$\lim_{x\to 0^+} \frac{1}{x} = \frac{1}{.1} = +\infty$

$x = .1$

$\lim_{x\to 0} \frac{1}{x} =$ DNE

$f(0) = \frac{1}{0}$ undef

not cont.

$f'(x) = -1x^{-2} = \frac{-1}{x^2}$

$f'(0)_- = -\frac{1}{0^2} =$ undef

$f'(0)_+ =$ undef

$f'(0) =$ DNE

not differentiable

If a function is differentiable at a point, then it is continuous at that point.

If a function is not continuous at a point, then it is not differentiable at that point.

1. $f'(1)$ exists. Find the values of a and b.

$$f(x)=\begin{cases} x^2 & \text{for } x<1 \\ ax+b & \text{for } x\geq 1 \end{cases}$$

$$f'(x)=\begin{cases} 2x \\ a \end{cases}$$

$x^2=ax+b$
$1^2=a(1)+b$
$1=a+b$

$2x=a$
$2(1)=a$
$\boxed{a=2}$

$1=2+b$
$-2 \quad -2$
$\boxed{b=-1}$

2. $f(x)$ is continuous and differentiable. Find the values of a and b.

$$f(x)=\begin{cases} ax^3-4x & \text{for } x\leq 1 \\ bx^2+2 & \text{for } x>1 \end{cases}$$

$$f'(x)=\begin{cases} 3ax^2-4 \\ 2bx \end{cases}$$

$ax^3-4x=bx^2+2$
$a(1)^3-4(1)=b(1)^2+2$
$a-4=b+2$
$-b+4 \quad -b \quad +4$

$a-b=6$
$+b \quad +b$
$a=b+6$

$3ax^2-4=2bx$
$3a(1)^2-4=2b(1)$
$3a-4=2b$

$3(b+6)-4=2b$
$3b+18-4=2b$
$3b+14=2b$
$-3b \quad -3b$
$14=-1b$
$\boxed{b=-14}$

$a=b+6$
$a=-14+6$
$\boxed{a=-8}$

3. Determine if $f(x)$ is continuous and differentiable at $x = 2$.

$$f(x) = \begin{cases} x^2 + 4 & \text{for } x < 2 \\ 3x + 2 & \text{for } x \geq 2 \end{cases}$$

$$f'(x) = \begin{cases} 2x & x < 2 \\ 3 & x > 2 \end{cases}$$

$\lim_{x \to 2^-} 2^2 + 4 = 4 + 4 = 8$

$f'_-(2) = 2(2) = 4$

$\lim_{x \to 2^+} 3(2) + 2 = 6 + 2 = 8$

$f'_+(2) = 3$

$\lim_{x \to 2} 8$

$f'(2) = \text{DNE}$

$f(2) = 3(2) + 2 = 8$

$\boxed{\text{not differentiable}}$

$\boxed{\text{continuous}}$