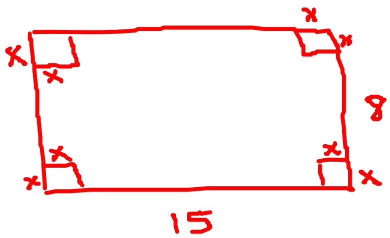
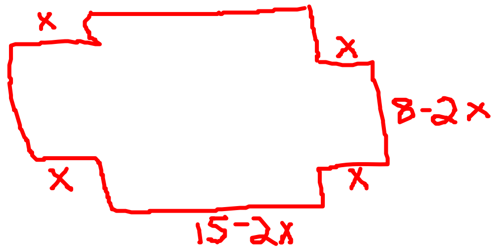
Optimization Problems

1. Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides.





$L = 15-2x$

$W = 8-2x$

$H = x$

$V = LWH$

$V = (15-2x)(8-2x)(x)$

$V = (120-30x-16x+4x^2)(x)$

$V = 120x-46x^2+4x^3$

$V' = 120-92x+12x^2$

$\frac{12x^2}{4}-\frac{92x}{4}+\frac{120}{4}=\frac{0}{4}$

$3x^2-23x+30=0$

$(3x-5)(x-6)=0$

$3x-5=0 \qquad x-6=0$

$3x=5 \qquad \cancel{x=6}$

$\boxed{x=5/3}$

$V'' = -92+24x$

$V''(5/3) = -92+24\left(\frac{5}{3}\right)$

$V''(5/3) = -92+40$

$V''(5/3) = -52$

$\cap$

$L = 15-2\left(\frac{5}{3}\right)$

$= \frac{3\cdot 15}{3\cdot 1}-\frac{10}{3} = \frac{45}{3}-\frac{10}{3}$

$= \frac{35}{3}$

$W = 8-2\left(\frac{5}{3}\right)$

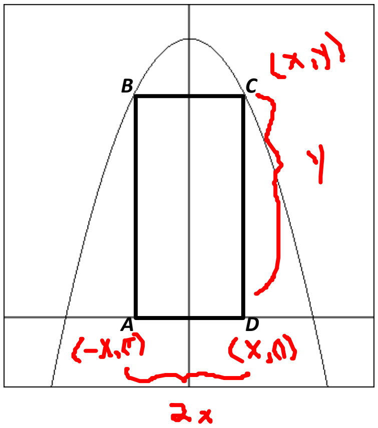
$= \frac{3\cdot 8}{3\cdot 1}-\frac{10}{3} = \frac{24}{3}-\frac{10}{3}$

$= \frac{14}{3}$

$H = \frac{5}{3}$

max. Vol. $= \left(\frac{35}{3}\right)\left(\frac{14}{3}\right)\left(\frac{5}{3}\right) = \boxed{\frac{2450}{27}\ in^3}$

2. Rectangle $ABCD$ with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the x-axis as shown in the figure below. Find the x and y coordinates of C so that the area of rectangle $ABCD$ is a maximum.



$A = lw$

$A = 2x \cdot y$

$A = 2x(-4x^2+4)$

$A = -8x^3 + 8x$

$A' = -24x^2 + 8$

$-24x^2 + 8 = 0$

$\quad -8 \quad -8$

$\dfrac{-24x^2}{-24} = \dfrac{-8}{-24}$

$\sqrt{x^2} = \sqrt{\dfrac{1}{3}}$

$x = \pm \dfrac{1}{\sqrt{3}}$

$x = \dfrac{1}{\sqrt{3}}$

$A'' = -48x$

$A''\left(\dfrac{1}{\sqrt{3}}\right) = -48 \cdot \dfrac{1}{\sqrt{3}}$

$= -\dfrac{48}{\sqrt{3}}$

$y = -4x^2 + 4$

$x = \dfrac{1}{\sqrt{3}}$

$y = -4\left(\dfrac{1}{\sqrt{3}}\right)^2 + 4$

$= -4 \cdot \dfrac{1}{3} + 4$

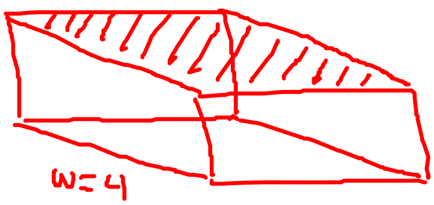
$= -\dfrac{4}{3} + \dfrac{4}{1} \cdot \dfrac{3}{3} = -\dfrac{4}{3} + \dfrac{12}{3}$

$= \dfrac{8}{3}$

$\boxed{C\left(\dfrac{1}{\sqrt{3}}, \dfrac{8}{3}\right) \text{ OR } \left(\dfrac{\sqrt{3}}{3}, \dfrac{8}{3}\right)}$

$\dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

3. A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs $10 per square meter for the base and $5 per square meter for the sides, what is the cost of the least expensive tank?



$V = 36\ m^3$

$V = LWH$

$36 = L \cdot 4 \cdot H$

$\frac{9}{L} = \frac{LH}{L}$

$H = \frac{9}{L}$

$SA = \cancel{2}LW + 2WH + 2LH$

$SA = LW + 2WH + 2LH$

$SA = L(4) + 2(4)\left(\frac{9}{L}\right) + 2\cancel{L}\left(\frac{9}{\cancel{L}}\right)$

$SA = 4L + 72L^{-1} + 18$

$\longrightarrow SA' = 4 - 72L^{-2}$

$SA' = \frac{L^2 \cdot 4}{L^2 \cdot 1} - \frac{72}{L^2}$ $\quad LCD = L^2$

$SA' = \frac{4L^2 - 72}{L^2}$

$4L^2 - 72 = 0$

$\quad +72 \quad +72$

$\frac{4L^2}{4} = \frac{72}{4}$

$\sqrt{L^2} = \sqrt{18}$

$L = 3\sqrt{2}$

$SA'' = 144L^{-3} = \frac{144}{L^3}$

$S''(3\sqrt{2}) = \frac{144}{(3\sqrt{2})^3}$ pos.

$L = 3\sqrt{2} \quad W = 4$

$H = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

$SA = LW + 2LH + 2WH$

$= (3\sqrt{2})(4)(10) + \cancel{2}(3\sqrt{2})\left(\frac{3\sqrt{2}}{\cancel{2}}\right)(5)$

$+ \cancel{2}(4)\left(\frac{3\sqrt{2}}{\cancel{2}}\right)(5)$

$= 120\sqrt{2} + 90 + 60\sqrt{2}$

$= \boxed{\$344.56}$

4. The sum of two non-negative numbers is 20. Find the numbers if:

1st: x 2nd: y

$\underline{x + y = 20}$

a) their product is to be as large as possible.

$P = xy$

$P = x(20 - x)$

$P = 20x - x^2$

$P' = 20 - 2x$

$20 - 2x = 0$

$-20 \quad -20$

$\frac{-2x}{-2} = \frac{-20}{-2}$

$x = 10$

$P'' = -2$

$x + y = 20$

$-x \quad -x$

$y = 20 - x$

$y = 20 - 10$

$y = 10$

$\boxed{x = 10 \quad y = 10}$

b) the sum of their squares is to be as small as possible.

$S = x^2 + y^2$

$S = x^2 + (20 - x)^2$

$S' = 2x + 2(20 - x)'(-1)$

$S' = 2x - 2(20 - x)$

$S' = 2x - 40 + 2x$

$S' = 4x - 40$

$4x - 40 = 0$

$+40 \quad +40$

$4x = 40$

$x = 10$

$S'' = 4$

$x + y = 20$

$-x \quad -x$

$y = 20 - x$

$y = 20 - 10$

$y = 10$

$\boxed{x = 10 \quad y = 10}$

c) one number plus the square root of the other is to be as large as possible.

$x + y = 20$

$y = 20 - x$

$S = x + \sqrt{y}$

$S = x + \sqrt{20-x}$

$S = x + (20-x)^{\frac{1}{2}}$

$\rightarrow \quad S' = 1 + \frac{1}{2}(20-x)^{-\frac{1}{2}}(-1)$

$S' = \frac{1}{1} \cdot \frac{2\sqrt{20-x}}{2\sqrt{20-x}} - \frac{1}{2\sqrt{20-x}}$ $\quad$ LCD $= 2\sqrt{20-x}$

$S' = \frac{2\sqrt{20-x} - 1}{2\sqrt{20-x}}$

$\underline{S' = 0}$

$2\sqrt{20-x} - 1 = 0$

$\quad +1 \quad +1$

$\frac{2\sqrt{20-x}}{2} = \frac{1}{2}$

$(\sqrt{20-x})^2 = \left(\frac{1}{2}\right)^2$

$20 - x = \frac{1}{4}$

$-20 \qquad -20$

$-x = -19\,3/4$

$x = 19\,3/4$

$S' = 1 - \frac{1}{2}(20-x)^{-\frac{1}{2}}$

$S'' = \frac{1}{4}(20-x)^{-3/2}(-1)$

$S'' = \frac{-1}{4(20-x)^{3/2}}$

$S''(19\,3/4) = \frac{-1}{4(20-19\,3/4)^{3/2}}$

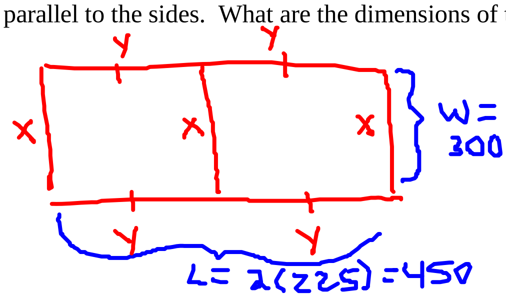
$= \text{Neg}$ $\frown$

$y = 20 - x$

$y = 20 - 19\,3/4$

$$\boxed{\begin{array}{l} y = 1/4 \\ x = 19\,3/4 \end{array}}$$

5. A farmer wishes to enclose a rectangular field by a fence and then divide it down the middle by another fence parallel to the sides. What are the dimensions of the largest area that can be enclosed with 1800 feet of fencing.



$$3x + 4y = 1800$$
$$-3x \qquad -3x$$

$$\frac{4y}{4} = \frac{1800}{4} - \frac{3x}{4}$$

$$y = 450 - \frac{3}{4}x$$

$$y = 450 - \frac{3}{\cancel{4}_1}(\cancel{300}^{75})$$

$$y = 450 - 225$$

$$y = 225$$

$$A = LW$$

$$A = 2y \cdot x$$

$$A = 2x\left(450 - \frac{3}{4}x\right)$$

$$A = 900x - \frac{6}{4}x^2$$

$$A = 900x - \frac{3}{2}x^2$$

$$A' = 900 - 3x$$

$$900 - 3x = 0$$
$$-900 \qquad -900$$

$$\frac{-3x}{-3} = \frac{-900}{-3}$$

$$x = 300$$

$$A'' = -3$$

$$\boxed{\begin{array}{l} L = 450 \\ W = 300 \end{array}}$$