Finding Volumes of
Solids Using the
Washer Method

Radius of outer function

Radius of inner function

Vertical Washers: Volume $= \int_a^b \pi\left(R^2(x) - r^2(x)\right)dx$

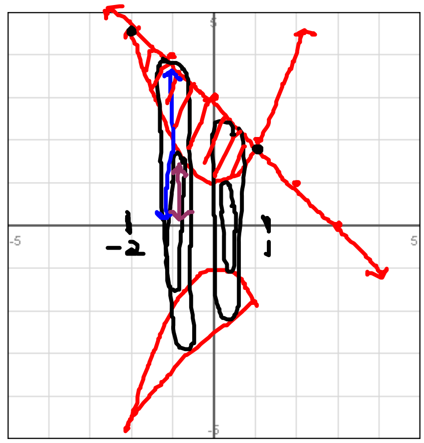
Horizontal Washers: Volume $= \int_c^d \pi\left(R^2(y) - r^2(y)\right)dy$

1. The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the $x-axis$ to generate a solid. Find the volume.

$$\int_a^b \pi\left[R^2(x) - r^2(x)\right]dx$$

$R = -x+3-0$
$= -x+3$

$r = x^2+1-0$
$= x^2+1$



$x^2+1 = -x+3$
$x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$
$x = -2 \quad x = 1$

$$\int_{-2}^{1} \pi\left[(-x+3)^2 - (x^2+1)^2\right]dx$$

$(-x+3)(-x+3)$
$x^2 - 3x - 3x + 9$
$x^2 - 6x + 9$

$(x^2+1)(x^2+1)$
$x^4 + x^2 + x^2 + 1$
$x^4 + 2x^2 + 1$

$x^2 - 6x + 9 - (x^4 + 2x^2 + 1)$
$x^2 - 6x + 9 - x^4 - 2x^2 - 1$
$-x^4 - x^2 - 6x + 8$

$$\int_{-2}^{1} \pi(-x^4 - x^2 - 6x + 8)dx = \pi\left(\frac{-x^5}{5} - \frac{x^3}{3} - \frac{6x^2}{2} + 8x\right)\Bigg|_{-2}^{1}$$

$$\pi\left[\left(-\frac{1}{5} - \frac{1}{3} - 3 + 8\right) - \left(\frac{32}{5} + \frac{8}{3} - 12 - 16\right)\right] = \pi\left[\frac{3\cdot 1}{3\cdot 5} - \frac{1\cdot 5}{3\cdot 5} + \frac{5\cdot 15}{1\cdot 15} - \frac{32\cdot 3}{5\cdot 3} - \frac{8\cdot 5}{3\cdot 5} + \frac{28\cdot 15}{1\cdot 15}\right]$$

LCD=15

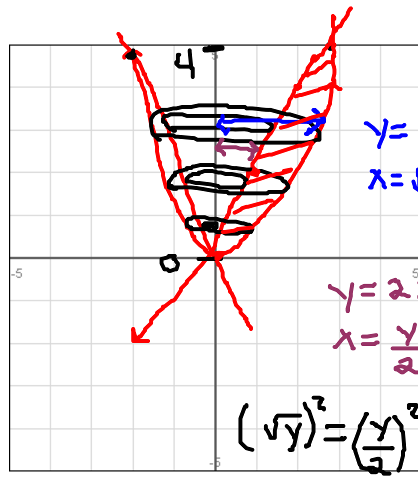
$$\pi\left[\frac{-3-5+75-96-40+420}{15}\right] = \pi \cdot \frac{351 \div 3}{15 \div 3} = \boxed{\frac{117\pi}{5}}$$

2. The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the $y-axis$ to generate a solid. Find the volume.

$$\int_c^d \pi\left(R^2(y) - r^2(y)\right)dy$$

$R = \sqrt{y} - 0$
$= \sqrt{y}$

$r = \frac{y}{2} - 0$
$= \frac{y}{2}$



$y = x^2$
$x = \sqrt{y}$

$y = 2x$
$x = \frac{y}{2}$

$(\sqrt{y})^2 = \left(\frac{y}{2}\right)^2$

$\frac{y}{1} = \frac{y^2}{4}$

$y^2 = 4y$

$y^2 - 4y = 0$

$y(y-4) = 0$

$y = 0 \quad y = 4$

$$\int_0^4 \pi\left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right]dy$$

$$\int_0^4 \pi\left[y - \frac{y^2}{4}\right]dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]\Bigg|_0^4$$

$$\pi\left[\left(8 - \frac{64}{12}\right) - \left(\frac{0}{2} - \frac{0}{12}\right)\right] = \pi\left(\frac{12\cdot 8}{12\cdot 1} - \frac{64}{12}\right) = \pi\left(\frac{96-64}{12}\right)$$

LCD = 12

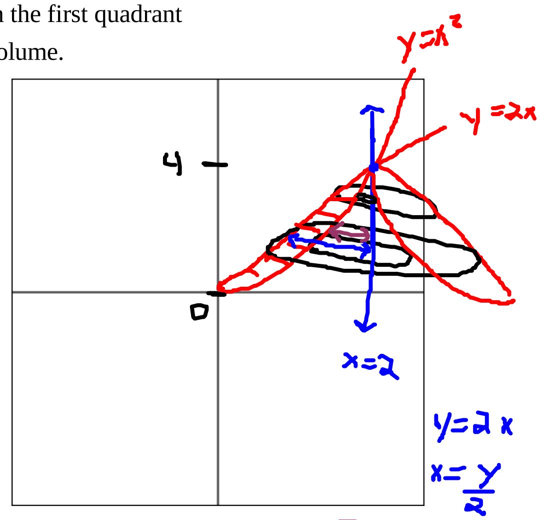
$$= \pi\,\frac{32 \div 4}{12 \div 4} = \boxed{\frac{8\pi}{3}}$$

3. The region between the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the line $x = 2$ to generate a solid. Find the volume.

$y = x^2 = 2^2 = 4$

$y = 2x = 2 \cdot 2 = 4$

$(2, 4)$



$y = 2x$

$x = \frac{y}{2}$

$y = x^2$

$x = \sqrt{y}$

$\int_c^d \pi \left[R^2(y) - r^2(y) \right] dy \qquad R = 2 - \frac{y}{2}$

$r = 2 - \sqrt{y}$

$\int_0^4 \pi \left[\left(2 - \frac{y}{2}\right)^2 - \left(2 - \sqrt{y}\right)^2 \right]$

$\left(2 - \frac{y}{2}\right)\left(2 - \frac{y}{2}\right) \qquad (2 - \sqrt{y})(2 - \sqrt{y})$

$4 - y - y + \frac{y^2}{4} \qquad 4 - 2\sqrt{y} - 2\sqrt{y} + y$

$4 - 4\sqrt{y} + y$

$4 - 2y + \frac{y^2}{4}$

$4 - 2y + \frac{y^2}{4} - (4 - 4\sqrt{y} + y)$

$\cancel{4} - 2y + \frac{y^2}{4} - \cancel{4} + 4\sqrt{y} - y$

$-3y + \frac{y^2}{4} + 4\sqrt{y}$

$\int_0^4 \pi \left[-3y + \frac{y^2}{4} + 4y^{\frac{1}{2}} \right] dy = \pi \left[\frac{-3y^2}{2} + \frac{y^3}{12} + \frac{4y^{3/2}}{3/2} \right] \Big|_0^4$

$\pi \left[\frac{-3y^2}{2} + \frac{y^3}{12} + \frac{8}{3} y^{3/2} \right] \Big|_0^4 = \pi \left[\left(-\frac{48}{2} + \frac{64}{12} + \frac{8}{3} \cdot 8 \right) - \left(\frac{0}{2} + \frac{0}{12} + \frac{8}{3}(0) \right) \right]$

$\pi \left[-\frac{24 \cdot 12}{1 \cdot 12} + \frac{64}{12} + \frac{64 \cdot 4}{3 \cdot 4} \right] = \pi \left[\frac{-288 + 64 + 256}{12} \right] = \pi \cdot \frac{32}{12}$

LCD = 12

$\pi \cdot \frac{32 \div 4}{12 \div 4} = \boxed{\frac{8\pi}{3}}$