Arc Length

If a function is continuous on $[a,b]$ then the length of the curve from a to b is:

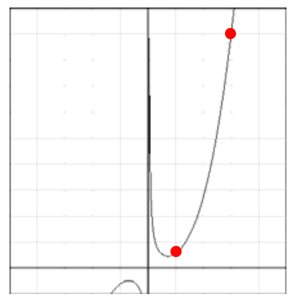
$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

If a function is continuous on $[c,d]$ then the length of the curve from c to d is:

$$L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

Directions: Find the length of each curve.

1. $y=\frac{x^3}{3}+\frac{1}{4x},\ 1\le x\le 3$



$$y=\frac{1}{3}x^3+\frac{1}{4}x^{-1}$$

$$y'=x^2-\frac{1}{4}x^{-2}=x^2-\frac{1}{4x^2}=\frac{4x^4-1}{4x^2}$$

$$(y')^2=\left(\frac{4x^4-1}{4x^2}\right)^2=\frac{16x^8-8x^4+1}{16x^4}$$

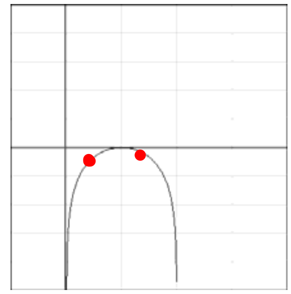
$$\int_1^3\sqrt{1+\frac{16x^8-8x^4+1}{16x^4}}\,dx=\int_1^3\sqrt{\frac{16x^4+16x^8-8x^4+1}{16x^4}}\,dx=\int_1^3\sqrt{\frac{16x^8+8x^4+1}{16x^4}}\,dx$$

$$\int_1^3\sqrt{\frac{(4x^4+1)^2}{16x^4}}\,dx=\int_1^3\frac{4x^4+1}{4x^2}\,dx=\int_1^3\frac{4x^4}{4x^2}+\frac{1}{4x^2}\,dx=\int_1^3 x^2+\frac{1}{4}x^{-2}\,dx$$

$$\frac{x^3}{3}-\frac{1}{4}x^{-1}=\frac{x^3}{3}-\frac{1}{4x}\Bigg]_1^3=\left(\frac{27}{3}-\frac{1}{12}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)=9-\frac{1}{12}-\frac{1}{3}+\frac{1}{4}$$

$$\frac{108-1-4+3}{12}=\frac{106}{12}=\boxed{\frac{53}{6}\text{ OR }8.8\overline{3}}$$

2. $y = \ln(\sin x),\ \frac{\pi}{4} \le x \le \frac{2\pi}{3}$



$y' = \frac{1}{\sin x} \cdot \cos x = \cot x$

$(y')^2 = \cot^2 x$

$\int_{\pi/4}^{2\pi/3} \sqrt{\underbrace{1+\cot^2 x}_{\csc^2 x}}\, dx$

$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$

$1 + \cot^2 x = \csc^2 x$

$\int_{\pi/4}^{2\pi/3} \sqrt{\csc^2 x}\, dx = \int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \csc x\, dx = \int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \csc x \cdot \frac{\csc x - \cot x}{\csc x - \cot x}\, dx$

$\int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\, dx = \int_{\pi/4}^{\frac{2\pi}{3}} \frac{1}{u}\, du = \ln|u| = \ln|\csc x - \cot x| \Big]_{\frac{\pi}{4}}^{\frac{2\pi}{3}}$

$u = \csc x - \cot x$

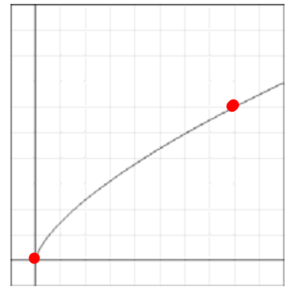
$du = -\csc x \cot x + \csc^2 x\, dx$

$\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln\left|\sqrt{2} - 1\right| = \ln \frac{3}{\sqrt{3}} - \ln(\sqrt{2}-1) = \ln\left(\frac{3}{\sqrt{3}} \div \frac{\sqrt{2}-1}{1}\right)$

$\frac{3}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}-1} = \frac{3}{\sqrt{6}-\sqrt{3}} \cdot \frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}+\sqrt{3}} = \frac{3(\sqrt{6}+\sqrt{3})}{6-3} = \frac{\cancel{3}(\sqrt{6}+\sqrt{3})}{\cancel{3}} = \sqrt{6}+\sqrt{3}$

$\boxed{\ln(\sqrt{6}+\sqrt{3}) \text{ OR } 1.43068}$

3. $y=\frac{3}{2}x^{\frac{2}{3}},\ [0,8]$



$y' = x^{-\frac{1}{3}} = \frac{1}{x^{1/3}}$

$(y')^2 = \left(\frac{1}{x^{1/3}}\right)^2 = \frac{1}{x^{2/3}}$

$\int_0^8 \sqrt{1+\frac{1}{x^{2/3}}}\,dx = \int_0^8 \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx = \int_0^8 \frac{1}{x^{1/3}}\sqrt{x^{2/3}+1}\,dx$

$u = x^{2/3}+1$

$du = \frac{2}{3}x^{-1/3}dx$

$\frac{3}{2}du = \frac{1}{x^{1/3}}dx$

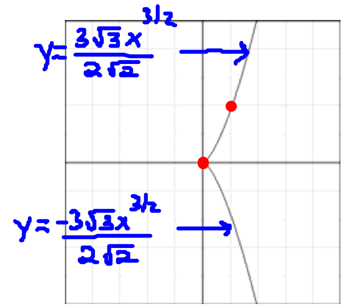
$\frac{3}{2}\int_0^8 \sqrt{u}\,du = \frac{3}{2}\int_0^8 u^{1/2}\,du = \frac{3}{2}\cdot\frac{2}{3}u^{3/2} = \left(x^{2/3}+1\right)^{3/2}\Big]_0^8$

$\left(8^{2/3}+1\right)^{3/2} - \left(0^{2/3}+1\right)^{3/2}\Big]_0^8 = 5^{3/2} - 1^{3/2} = \boxed{5\sqrt{5}-1 \text{ OR } 10.1803}$

$\sqrt{5}\cdot\sqrt{5}\cdot\sqrt{5}$

$5\sqrt{5}$

4. $27x^3 = 8y^2$, from $(0,0)$ to $\left(1, \frac{3\sqrt{6}}{4}\right)$



$$8y^2 = 27x^3$$

$$\sqrt{y^2} = \sqrt{\frac{27x^3}{8}}$$

$$y = \pm\sqrt{\frac{27x^3}{8}} = \pm \frac{3\sqrt{3}x^{3/2}}{2\sqrt{2}}$$

$$y' = \frac{3\sqrt{3}}{2\sqrt{2}} \cdot \frac{3}{2} x^{1/2} = \frac{9\sqrt{3}}{4\sqrt{2}} x^{1/2}$$

$$(y')^2 = \left(\frac{9\sqrt{3}}{4\sqrt{2}} x^{1/2}\right)^2 = \frac{81 \cdot 3}{16 \cdot 2} x = \frac{243}{32} x$$

$$\int_0^1 \sqrt{1 + \frac{243}{32}x}\, dx = \frac{32}{243}\int_0^1 \sqrt{u}\, du = \frac{32}{243}\int_0^1 u^{1/2}\, du = \frac{32}{243} \cdot \frac{2}{3} u^{3/2}$$

$$u = 1 + \frac{243}{32}x$$

$$du = \frac{243}{32}dx$$

$$\frac{32}{243}du = dx$$

$$\frac{64}{729}\left(1 + \frac{243}{32}x\right)^{3/2}\Big]_0^1$$

$$\frac{64}{729}\left[\left(1 + \frac{243}{32}\cdot 1\right)^{3/2} - \left(1 + \frac{243}{32}\cdot 0\right)^{3/2}\right]$$

$$\frac{64}{729}\left[\left(\frac{275}{32}\right)^{3/2} - 1^{3/2}\right] = \frac{64}{729}\left[\left(\frac{5\sqrt{11}}{4\sqrt{2}}\right)^3 - 1\right] = \frac{64}{729}\left[\frac{1375\sqrt{11}}{128\sqrt{2}} - 1\right]$$

$$5\sqrt{11} \cdot 5\sqrt{11} \cdot 5\sqrt{11} \qquad 4\sqrt{2} \cdot 4\sqrt{2} \cdot 4\sqrt{2}$$

$$25 \cdot 11 \cdot 5\sqrt{11} \qquad 16 \cdot 2 \cdot 4\sqrt{2}$$

$$1375\sqrt{11} \qquad 128\sqrt{2}$$

$$\frac{1375\sqrt{11}}{128\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{1375\sqrt{22}}{256} \qquad \frac{64}{729}\left[\frac{1375\sqrt{22}}{256} - 1\right]$$

$$\frac{\cancel{64}^{\,1}}{729}\left[\frac{1375\sqrt{22} - 256}{\cancel{256}\ 4}\right] = \boxed{\frac{1375\sqrt{22} - 256}{2916} \text{ OR } 2.12391}$$

OR Integrate with respect to y

$$\frac{27x^3}{27}=\frac{8y^2}{27} \qquad (0,0) \text{ to } \left(1,\frac{3\sqrt{6}}{4}\right)$$

$$\sqrt[3]{x^3}=\sqrt[3]{\frac{8y^2}{27}}$$

$$x=\sqrt[3]{\frac{8y^2}{27}}=\frac{2y^{2/3}}{3}=\frac{2}{3}y^{2/3}$$

$$x'=\frac{4}{9}y^{-1/3}=\frac{4}{9y^{1/3}}$$

$$(x')^2=\left(\frac{4}{9y^{1/3}}\right)^2=\frac{16}{81y^{2/3}}$$

$$\int_0^{\frac{3\sqrt{6}}{4}}\sqrt{1+\frac{16}{81y^{2/3}}}\,dy=\int_0^{\frac{3\sqrt{6}}{4}}\sqrt{\frac{81y^{2/3}+16}{81y^{2/3}}}\,dy=\int_0^{\frac{3\sqrt{6}}{4}}\frac{1}{9y^{1/3}}\sqrt{81y^{2/3}+16}\,dy$$

$$u=81y^{2/3}+16$$

$$du=54y^{-1/3}dy$$

$$\frac{1}{54}du=\frac{1}{y^{1/3}}dy$$

$$\frac{1}{54}\cdot\frac{1}{9}\int_0^{\frac{3\sqrt{6}}{4}}\sqrt{u}\,du=\frac{1}{486}\int_0^{\frac{3\sqrt{6}}{4}}u^{1/2}\,du=\frac{1}{\cancel{486}_{243}}\cdot\frac{\cancel{2}^{1}}{3}u^{3/2}=\frac{1}{729}\left(81y^{2/3}+16\right)^{3/2}\Bigg]_0^{\frac{3\sqrt{6}}{4}}$$

$$\frac{1}{729}\left\{\left[81\cdot\left(\frac{3\sqrt{6}}{4}\right)^{2/3}+16\right]^{3/2}-\left[81\cdot 0^{2/3}+16\right]^{3/2}\right\}$$

$$\frac{3\sqrt{6}}{4}\cdot\frac{3\sqrt{6}}{4}=\frac{9\cdot\cancel{6}^{3}}{\cancel{16}_{8}}=\frac{27}{8}$$

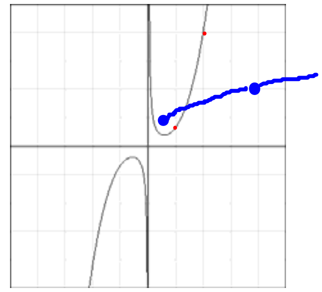
$$\sqrt[3]{\frac{27}{8}}=\frac{3}{2}$$

$$\frac{1}{729}\left[\left(81\cdot\frac{3}{2}+16\right)^{3/2}-16^{3/2}\right]=\frac{1}{729}\left[\left(\frac{275}{2}\right)^{3/2}-16^{3/2}\right]$$

$$\frac{1}{729}\left[\left(\frac{5\sqrt{11}}{\sqrt{2}}\right)^3-64\right]=\frac{1}{729}\left[\frac{1375\sqrt{11}}{2\sqrt{2}}-64\right]=\frac{1}{729}\left[\frac{1375\sqrt{22}}{4}-64\right]$$

$$\frac{1}{729}\left[\frac{1375\sqrt{22}-256}{4}\right]=\boxed{\frac{1375\sqrt{22}-256}{2916} \text{ OR } 2.12391}$$

5. $x=\frac{y^3}{2}+\frac{1}{6y},\ 1\le y\le 3$



$$x=\frac{1}{2}y^3+\frac{1}{6}y^{-1}$$

$$x'=\frac{3}{2}y^2-\frac{1}{6}y^{-2}=\frac{3y^2}{2}-\frac{1}{6y^2}=\frac{9y^4-1}{6y^2}$$

$$(x')^2=\left(\frac{9y^4-1}{6y^2}\right)^2=\frac{81y^8-18y^4+1}{36y^4}$$

$$\int_1^3\sqrt{1+\frac{81y^8-18y^4+1}{36y^4}}\,dy=\int_1^3\sqrt{\frac{36y^4+81y^8-18y^4+1}{36y^4}}\,dy$$

$$\int_1^3\sqrt{\frac{81y^8+18y^4+1}{36y^4}}\,dy=\int_1^3\sqrt{\frac{(9y^4+1)^2}{36y^4}}\,dy=\int_1^3\frac{9y^4+1}{6y^2}\,dy$$

$$\int_1^3\frac{9y^4}{6y^2}+\frac{1}{6y^2}\,dy=\int_1^3\frac{3}{2}y^2+\frac{1}{6}y^{-2}\,dy=\frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{3}}y^3-\frac{1}{6}y^{-1}\Big]_1^3$$

$$\frac{y^3}{2}-\frac{1}{6y}\Big]_1^3=\left(\frac{27}{2}-\frac{1}{18}\right)-\left(\frac{1}{2}-\frac{1}{6}\right)=\frac{27}{2}-\frac{1}{18}-\frac{1}{2}+\frac{1}{6}$$

$$\frac{243-1-9+3}{18}=\frac{236}{18}=\boxed{\frac{118}{9}\ \text{OR}\ 13.\overline{1}}$$