Trigonometric Substitutions

Expression	Substitution
a^2+u^2	$u=a\tan\theta$
a^2-u^2	$u=a\sin\theta$
u^2-a^2	$u=a\sec\theta$

Directions: Evaluate each integral.

1. $\int \frac{1}{\sqrt{x^2+16}}\,dx = \int \frac{1}{\sqrt{16\tan^2\theta+16}}\cdot 4\sec^2\theta\,d\theta = \int \frac{4\sec^2\theta}{\sqrt{16(\tan^2\theta+1)}}\,d\theta$

$u=x \quad a=4$

$u=a\tan\theta$

$*\ x=4\tan\theta$

$dx=4\sec^2\theta\,d\theta$

$$= \int \frac{4\sec^2\theta}{\sqrt{16\sec^2\theta}}\,d\theta = \int \frac{4\sec^2\theta}{4|\sec\theta|}\,d\theta = \int \sec\theta\,d\theta$$

$$= \int \sec\theta \cdot \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

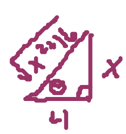
$u=\sec\theta+\tan\theta$

$du=\sec\theta\tan\theta+\sec^2\theta\,d\theta$

$$= \int \frac{1}{u}\,du = \ln|u| = \ln|\sec\theta+\tan\theta| = \ln\left|\frac{\sqrt{x^2+16}}{4}+\frac{x}{4}\right|$$

$*\ x=4\tan\theta$

$\tan\theta=\frac{x}{4}$



$$\boxed{= \ln\left|\frac{\sqrt{x^2+16}+x}{4}\right| + C}$$

2. $\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\, d\theta = \int \frac{4\sin^2\theta \cdot 2\cos\theta}{\sqrt{4(1-\sin^2\theta)}}\, d\theta$

$u = x \quad a = 2$

$u = a\sin\theta$

$* \; x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$= \int \frac{8\sin^2\theta\cos\theta}{\sqrt{4\cos^2\theta}}\, d\theta = \int \frac{8\sin^2\theta\cos\theta}{2\,|\cos\theta|}\, d\theta = \int 4\sin^2\theta\, d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \int 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta = 2\int 1-\cos 2\theta\, d\theta = 2\int 1\, d\theta - 2\int \cos 2\theta\, d\theta$$

$u = 2\theta$

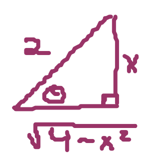
$du = 2\,d\theta$

$\frac{1}{2}du = d\theta$

$$= 2\int 1\, d\theta - \int \cos u\, du = 2\theta - \sin u = 2\theta - \sin 2\theta$$

$* \; x = 2\sin\theta$

$\sin\theta = \frac{x}{2}$



$\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\theta - 2\sin\theta\cos\theta = 2\arcsin\frac{x}{2} - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}$$

$$\boxed{= 2\arcsin\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} + C}$$

3. $\int \frac{1}{x\sqrt{4x^2+9}}\,dx = \int \frac{1}{\frac{3}{2}\tan\theta \sqrt{4\left(\frac{9}{4}\tan^2\theta\right)+9}} \cdot \frac{3}{2}\sec^2\theta \, d\theta$

$u = 2x \quad a = 3$

$u = a\tan\theta$

$* \; 2x = 3\tan\theta$

$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta \, d\theta$

$$= \int \frac{\sec^2\theta}{\tan\theta\sqrt{9(\tan^2\theta+1)}}\,d\theta = \int \frac{\sec^2\theta}{\tan\theta\sqrt{9\sec^2\theta}}\,d\theta = \int \frac{\sec^2\theta}{\tan\theta \cdot 3\sec\theta}\,d\theta$$

$$= \frac{1}{3}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{3}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\,d\theta = \frac{1}{3}\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{3}\int \csc\theta \, d\theta$$

$$= \frac{1}{3}\int \csc\theta \cdot \frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}\,d\theta = \frac{1}{3}\int \frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta - \cot\theta}\,d\theta$$

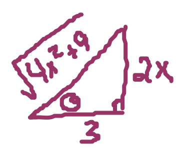
$u = \csc\theta - \cot\theta$

$du = -\csc\theta\cot\theta + \csc^2\theta \, d\theta$

$$= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| = \frac{1}{3}\ln\left|\csc\theta - \cot\theta\right|$$

$* \; 2x = 3\tan\theta$

$\tan\theta = \frac{2x}{3}$



$$= \frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}}{2x} - \frac{3}{2x}\right| = \boxed{\ln\sqrt[3]{\left|\frac{\sqrt{4x^2+9}-3}{2x}\right|} + C}$$

4. $\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\,dx = \int \frac{1}{(9\sec^2\theta - 9)^{3/2}} \cdot 3\sec\theta\tan\theta\, d\theta$

$u = x \quad a = 3$

$u = a\sec\theta$

$* \; x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\, d\theta$

$$= \int \frac{3\sec\theta\tan\theta}{\left[9(\sec^2\theta - 1)\right]^{3/2}}\,d\theta = \int \frac{3\sec\theta\tan\theta}{(9\tan^2\theta)^{3/2}}\,d\theta = \int \frac{3\sec\theta\tan\theta}{27\tan^3\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta \qquad \frac{\sec\theta}{\tan^2\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{1}{u^2}\,du = \frac{1}{9}\cdot\frac{-1}{u} = \frac{-1}{9\sin\theta}$$

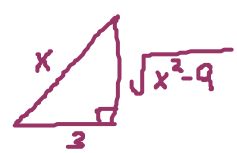
$u = \sin\theta$

$du = \cos\theta\, d\theta$

$* \; x = 3\sec\theta$

$\sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x}$



$$= \frac{-1}{9\left(\frac{\sqrt{x^2-9}}{x}\right)} = \frac{-1}{\frac{9\sqrt{x^2-9}}{x}} = \boxed{\frac{-x}{9\sqrt{x^2-9}} + C}$$