

Real Zeros of Polynomial Functions (Remainder and Factor Theorems, Rational Root Test and Upper/Lower Bounds)

Remainder Theorem - If a polynomial function $f(x)$ is divided by $x - k$, the remainder $f(k) = r$.

1. Use the Remainder Theorem to evaluate $f(2)$ in $f(x) = 2x^3 - 3x^2 + 8x - 7$.

$$\begin{array}{r} 2 \\ \hline 2 & -3 & 8 & -7 \\ & 4 & 2 & 20 \\ \hline 2 & 1 & 10 & \boxed{13} \\ \boxed{13} \end{array}$$

Check

$$\begin{aligned} f(2) &= 2(2)^3 - 3(2)^2 + 8(2) - 7 \\ &= 2(8) - 3(4) + 16 - 7 \\ &= 16 - 12 + 16 - 7 \\ &= \boxed{13} \end{aligned}$$

Factor Theorem - A polynomial function $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

2. Determine if $\underline{x-2}=0$ is a factor of $f(x) = 6x^4 - 11x^3 - 11x^2 + 14x + 8$.

$$\begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned}$$

$$\begin{array}{r} 2 \\ \hline 6 & -11 & -11 & 14 & 8 \\ & 12 & 2 & -18 & -8 \\ \hline 6 & 1 & -9 & -4 & \boxed{0} \end{array}$$

Since the remainder
 $= 0$, $x-2$ is a
 Factor of $f(x)$.

Rational Zero Test - If a polynomial function has integer coefficients then the possible rational zeros are of the form

$\frac{p}{q}$, where q represents the factors of the leading coefficient and p represents the factors of the constant.

3. List the possible rational zeros of $f(x)$.

a) $f(x) = x^3 + 3x^2 + x + 6$

$$\begin{array}{c} \boxed{1} \\ \boxed{3} \\ \hline \boxed{1} \end{array} \quad \begin{array}{c} \boxed{1} \\ \boxed{6} \\ \hline \boxed{1} \end{array}$$

$$p = \pm 1, \pm 2, \pm 3, \pm 6$$
$$q = \pm 1$$

$$\boxed{p/q = \pm 1, \pm 2, \pm 3, \pm 6}$$

b) $f(x) = 2x^4 - 3x^2 + 12$

$$\begin{array}{c} \boxed{1} \\ \boxed{2} \\ \hline \boxed{1} \end{array} \quad \begin{array}{c} \boxed{1} \\ \boxed{12} \\ \hline \boxed{2} \end{array}$$
$$\begin{array}{c} \boxed{1} \\ \boxed{6} \\ \hline \boxed{3} \end{array} \quad \begin{array}{c} \boxed{1} \\ \boxed{4} \\ \hline \boxed{3} \end{array}$$

$$p: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$
$$q: \pm 1, \pm 2$$

$$\boxed{\begin{array}{l} p: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2} \\ q: \pm \frac{3}{2}, \end{array}}$$

4. Find all the real zeros of each polynomial equation.

$$a) f(x) = \frac{6x^4 - 11x^3 - 11x^2 + 14x + 8}{x^2 - 1}$$

$$P: \pm 1, \pm 2, \pm 4, \pm 8$$

$$Q: \pm 1, \pm 2, \pm 3, \pm 6$$

$$\frac{P}{Q}: \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{8}{3}, \pm \frac{2}{3}, \pm \frac{1}{2} \\ \pm \frac{4}{3}, \pm \frac{8}{3}$$

$$\begin{array}{r} -1 \\ \hline 6 & -11 & -11 & 14 & 8 \\ & -6 & 17 & -6 & -8 \\ \hline 6x^3 & -17x^2 & 6x & 8 & 0 \end{array}$$

$$\begin{array}{r} 2 \\ \hline 6 & -17 & 6 & 8 \\ & 12 & -10 & -8 \\ \hline 6x^2 & -5x & -4 & 0 \end{array}$$

$$6x^2 - 5x - 4 = 0$$

$$(3x - 4)(2x + 1) = 0$$

$$3x - 4 = 0 \quad 2x + 1 = 0$$

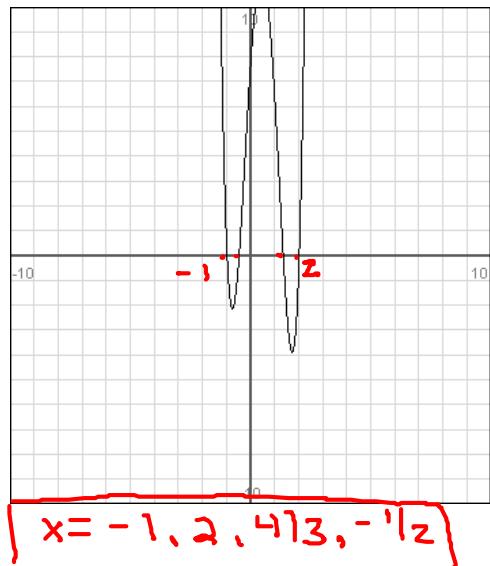
$$3x = 4$$

$$x = \frac{4}{3}$$

$$2x + 1 = 0$$

$$2x = -1$$

$$x = -\frac{1}{2}$$



$$b) f(x) = \frac{2x^4 - 19x^3 + 42x^2 + 9x - 54}{1 \cdot 2}$$

$\frac{q}{p} = \frac{-54}{2} = -27$
 $\frac{3 \cdot 18}{6 \cdot 9} = \frac{1}{1}$
 $D: \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm 27, \pm 54$
 $q: \pm 1, \pm 2$
 $P/q: \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm 27, \pm 54$
 $\pm 1|_2, \pm 3|_2, \pm 9|_2, \pm 27|_2$

$$1 \left| \begin{array}{ccccc} 2 & -19 & 42 & 9 & -54 \\ & 2 & -17 & 25 & 34 \\ \hline 2 & -17 & 25 & 34 & \boxed{-20} \end{array} \right. (1, -20)$$

$$2 \left| \begin{array}{ccccc} 2 & -19 & 42 & 9 & -54 \\ & 4 & -30 & 24 & 66 \\ \hline 2 & -15 & 12 & 33 & \boxed{12} \end{array} \right. (2, 12)$$

$$\frac{2x^2 - 10x - 12}{2} = 0$$

$$*3/2 \left| \begin{array}{ccccc} 2 & -19 & 42 & 9 & -54 \\ & 3 & -24 & 27 & 54 \\ \hline 2x^3 - 16x^2 + 18x + 36 & \boxed{0} \end{array} \right.$$

$$\rightarrow x^2 - 5x - 6 = 0$$

$$(x - 6)(x + 1) = 0$$

$$x - 6 = 0 \quad x + 1 = 0$$

$$x = 6 \quad x = -1$$

$$*3 \left| \begin{array}{ccccc} 2 & -16 & 18 & 36 \\ & 6 & -30 & -36 \\ \hline 2x^2 - 20x - 12 & \boxed{0} \end{array} \right.$$

Zeros: $3/2, 3, 6, -1$

Upper and Lower Bound Rule - If a polynomial function has real coefficients with a positive leading coefficient, when the polynomial is divided by $x - c$ using synthetic division,

If $c > 0$ and the last row is either positive or zero, then c is an upper bound.

If $c < 0$ and the last row alternates from positive to negative, then c is a lower bound.

5. Use synthetic division to determine if each value of x is an upper bound or a lower bound.

$$f(x) = x^4 - 4x^3 + 16x - 16$$

LC = 1

a) $x = 5$

$$\begin{array}{r} 5 \\ \hline 1 & -4 & 0 & 16 & -16 \\ & 5 & 5 & 25 & 205 \\ \hline & 1 & 1 & 5 & 41 & | 189 \end{array}$$

5 is an UB

b) $x = -3$

$$\begin{array}{r} -3 \\ \hline 1 & -4 & 0 & 16 & -16 \\ & -3 & 21 & -63 & 141 \\ \hline & 1 & -7 & 21 & -47 & | 125 \end{array}$$

-3 is a LB

